Aggregation Functions and Personal Utility Functions in General Insurance

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Abstract: The modeling of a utility function's forms is a very interesting part of modern decision making theory. We apply a basic concept of the personal utility theory on determination of minimal net and maximal gross annual premium in general insurance. We introduce specific values of gross annual premium on the basis of a personal utility function, which is determined empirically by a short personal interview. Moreover, we introduce a new approach to the creation of a personal utility function by a fictive game and an aggregation of specific values by mixture operators.

Keywords: Utility function; Expected utility; Mixture operator; General Insurance

1 Introduction

This paper was mainly inspired by the books Modern Actuarial Risk Theory [3] and Actuarial models – The Mathematics of Insurance [13]. The authors of the above-mentioned books assume utility functions as linear utility $u(w) = w$, quadratic utility $u(w) = -(a-w)^2$, power utility $u(w) = w^c$, etc. Lapin in [5] describes and explains an application of the utility function in decision making in a really interesting way. In this book also the generation of the utility function using information extracted from a personal interview is explained. A modern theoretical approach to the utility function is also described by Norstad in [8].

We can find a very interesting discussion about utility functions in [4]. An alternative approach to the determination of a utility function on the basis of the aggregation of specific utility values can be found in [18].

However, in real life people do not behave according to the theoretical utility functions. It is a psychological problem rather than a mathematical one. The seriousness and also the uncertainty of a respondent's answers depend on the situation, on the form of the asked questions, on the time which the respondents have, and on a lot of other psychological and social factors. In our paper we introduce the possibility of determining a personal utility function on the basis of a personal interview with virtual money.

Moreover, we recall and apply one type of aggregation operators [2], the so-called mixture operators – M_g, the generalized mixture operators – M_{g}, and the specially ordered generalized mixture operators – M_{g}' on the aggregation of so-called risk neutral points, see [6-7], [9-11], [14-16].

This paper is organized as follows: in Section 2 we recall the basic properties of utility functions and their applications in general insurance. In Section 3 we also recall mixture operators and their properties, namely the sufficient conditions of their non-decreasing-ness. In Section 4 we describe the personal utility function of our respondent who took part in our short interview. Using this function we calculate the maximal gross premium for a general insurance policy. In this section we also describe an alternative approach, where a personal utility function is determined through the result of a fictive game and theoretical utility functions. The resulting utility function is then used for the computation of the maximal gross premium. Moreover, we evaluate the minimal net annual premium by means of the theoretical utility function for the insurer. Finally, some conclusions and indications as to our next investigation about the mentioned topic are included.

2 Utility Functions

Individuals can have very different approaches to risk. A personal utility function can be used as a basis for describing them. In general, we can identify three basic personalities with respect to risk. The risk-averse individual, who accepts favorable gambles only, a risk seeker, or in other words risk-loving individual, who pays a premium for the privilege of participation in a gamble, and the risk-neutral individual, who considers the face value of money to be its true worth. Throughout most of their life people are typically risk averse. Only gambles with high expected payoff will be attractive to them. The risk-averse individual's marginal utility diminishes as the benefits increase, so that the risk-averse individual's utility function exhibits a decreasing positive slope as the level of monetary payoff becomes higher. Such a function is concave, see Figure 1.

The behavior of a risk-loving individual is opposite. The risk-loving individual prefers some gambles with negative expected monetary payoffs. Their marginal utility increases. Each additional euro provides a disproportionately greater sense of well-being. Thus, the slope of the risk-loving individual's utility function increases as the monetary change improves. This function is convex (see Fig. 2). The utility function for a risk-neutral individual is a straight line. The utility is equal to the utility of expected value. Risk-neutral individuals buy no casually insurance since the premium charge is greater than the expected loss. Risk-neutral behavior is typical for persons who are enormously wealthy.

Of course, a lot of people may be risk averse and risk loving at the same time, depending on the range of monetary values being considered, which can be illustrated using the behavior of the personal utility function of our respondent.

2.1 The Personal Utility Function

The fundamental proposition of the modern approach to utility is the possibility to obtain a numerical expression for individual preferences. As people usually have different approaches to risk, two persons faced with an identical decision may actually prefer different courses of action. In this section we will discuss utility as an alternative expression of payoff that reflects personal approaches.

Suppose that our respondent owns capital w, and that he values wealth by the utility function u. The next Theorem 1, or in other words Jensen`s inequality, describes the properties of the utility function and its expected value [3], (see also Figure 2). It can be written as follows.

Theorem 1 [3] (Jensen's inequality)

If $u(x)$ is a convex function and X is a random variable, then the expected utility is greater or equals to a utility value

$$E[u(X)] \geq u(E[X]) \tag{1}$$

with an equality if and only if $u(x)$ is linear with respect to X or $\operatorname{var}(X) = 0$.

From Jensen`s inequality and Figure 1 it follows that for a concave utility function it holds

$$E[u(w-X)] \leq u(E[w-X]) = u(w-E[X]). \tag{2}$$

In this case the decision maker is called risk averse. He prefers to pay a fixed amount $E[X]$ instead of a risk amount X.

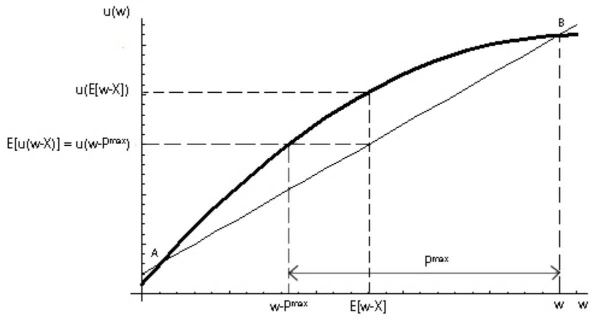

Figure 1
Concave utility function - risk averse approach

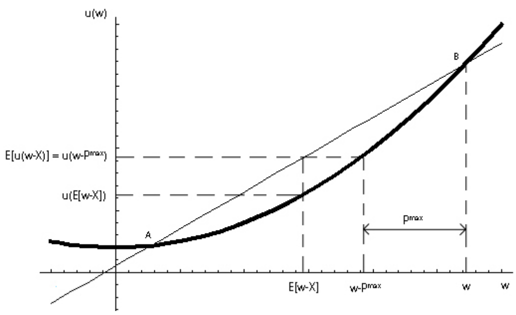

Figure 2
Convex utility function - risk loving approach

In the next part we illustrate whether to buy insurance or not by evaluating an individual's decision. Now suppose that our respondent has two alternatives, to buy insurance or not. Assume he is insured against a loss X for a premium P .

If he is insured, this means a certain alternative. This decision gives us the utility value $u(w-P)$.

If he is not insured, this means an uncertain alternative. In this case the expected utility is $E[u(w-X)]$.

From Jensen's inequality (2) we get

$$E[u(w-X)] \leq u(E[w-X]) = u(w-E[X]) = u(w-P). \tag{3}$$

Since a utility function u is a non-decreasing continuous function, this is equivalent to $P \leq P^{\max}$, where $P^{\max}$ denotes the maximum premium to be paid.

This so-called zero utility premium is a solution of the following utility equilibrium equation

$$E[u(w-X)] = u\left(w-P^{\max}\right). \tag{4}$$

The difference $\left(w-P^{\max}\right)$ is also called the certainty equivalent - CE . In $[3]$ the certainty equivalent is defined as follows.

Definition 1 The certainty equivalent is that payoff amount that the decision maker would be willing to receive in exchange for undergoing the actual uncertainty, taking into account its benefits and risks.

Remark 1 We recall that the expected utility is calculated by means of the well-known formula

$$E[u(X)] = \sum_{i=1}^{n} u(x_i)\cdot p_i\,, \tag{5}$$

where $X=(x_1,x_2,...,x_n)$ is a vector of the possible alternatives and p_i, for $i=1,2,...,n$, are respective probabilities.

Expected utilities can be calculated as function values of a linear function, which is assigned uniquely by points A and B, where point A represents the worst outcome and B the best outcome.

Remark 2 $[5]$ When possible monetary outcomes fall into the decision maker's range of risk averse, the following properties hold (see Figure 1):

1) Expected payoffs $EP = E[w-X]$ are greater than their counterpart certainty equivalent $CE = w-P^{\max}$.

2) Expected utilities $E[u(w-X)]$ will be less than the utility of the respective expected monetary payoff $u\left(w-P^{\max}\right)$.

3) Risk premiums $RP = EP-CE$ are positive.

If possible monetary outcomes fall into the decision maker's range of risk loving, the following properties hold (see Figure 2):

1) Expected payoffs $EP = E[w-X]$ are less than their counterpart certainty equivalent $CE = w-P^{\max}$.

2) Expected utilities $E[u(w-X)]$ will be greater than the utility of the respective expected monetary payoff $u\left(w-P^{\max}\right)$.

3) Risk premiums $RP = EP-CE$ are negative.

The insurer with a utility function U and capital W, with insurance of loss X for a premium P must satisfy the inequality

$$E[U(W+P-X)] \geq U(W), \tag{6}$$

and hence for the minimal accepted premium $P^{\min}$

$$U(W) = E\left[U\left(W+P^{\min}-X\right)\right]. \tag{7}$$

2.2 The Risk Aversion Coefficient

On the basis of equation (3) we can evaluate a risk aversion coefficient. Let μ and σ^2 be the mean and variance of loss X. Using the first terms in the Taylor expansion of the utility function u in $w-\mu$, we obtain

$$u(w-X) \approx u(w-\mu) + u'(w-\mu)\cdot(\mu-X) + \frac{1}{2}u''(w-\mu)\cdot(\mu-X)^2.$$

The expected utility from $u(w-X)$ is given by

$$\begin{aligned} &E[u(w-X)] \approx \\ &\approx E\left[u(w-\mu) + (\mu-X)\cdot u'(w-\mu) + \frac{1}{2}(\mu-X)^2\cdot u''(w-\mu)\right] \end{aligned}$$

After some processing we get

$$E[u(w-X)] \approx u(w-\mu) + \frac{1}{2}\sigma^2\cdot u''(w-\mu). \tag{8}$$

The Taylor expansion of the function on the right side of equation (3) is given by

$$u\left(w-P^{\max}\right) \approx u(w-\mu) + \left(\mu-P^{\max}\right)\cdot u'(w-\mu). \tag{9}$$

From the equality of equations (8) and (9) we have

$$u(w-\mu) + \frac{1}{2}\sigma^2\cdot u''(w-\mu) \approx u(w-\mu) + \left(\mu-P^{\max}\right)\cdot u'(w-\mu). \tag{10}$$

After some processing we get

$$P^{\max} \approx \mu - \frac{1}{2}\sigma^2\frac{u''(w-\mu)}{u'(w-\mu)}, \tag{11}$$

where a risk aversion coefficient $r(w)$ of the utility function u at a wealth $w-\mu$ is given by

$$r(w) = -\frac{u''(w-\mu)}{u'(w-\mu)} . \tag{12}$$

$$P^{\max} \approx \mu + \frac{1}{2} r(w-\mu) \cdot \sigma^2 . \tag{13}$$

From (13) you can see that, if the insured has greater risk aversion coefficient, then he is willing to pay greater premium.

3 Mixture Operators

In this part we review some mixture operators introduced in [6], [7], [9-11]. Suppose that each alternative ***x*** is characterized by a score vector $\boldsymbol{x}=(x_1,...x_n) \in [0,1]^n$, where $n \in N - \{1\}$ is the number of applied criteria. A mixture operator can be defined as follows:

Definition 2 A mixture operator $M_g : [0,1]^n \to [0,1]$ is the arithmetic mean weighted by a continuous weighting function $g : [0,1] \to]0, \infty[$ given by

$$M_g(x_1,...,x_n) = \frac{\sum_{i=1}^{n} g(x_i) \cdot x_i}{\sum_{i=1}^{n} g(x_i)} , \tag{14}$$

where $(x_1,...,x_n)$ is an input vector.

Observe that due to the continuity of weighting function g, each mixture operator M_g is continuous. Evidently, M_g is an idempotent operator, [2], [6], [9-10]. Note that sometimes different continuous weighting functions are applied for different criteria score, which leads to a generalized mixture operator, see [6], [9-10].

Definition 3 A generalized mixture operator $M_{\mathbf{g}}$**:** $[0,1]^n \to [0,1]$ is given by

$$M_g(x_1,...,x_n) = \frac{\sum_{i=1}^{n} g_i(x_i) \cdot x_i}{\sum_{i=1}^{n} g_i(x_i)} , \tag{15}$$

where $(x_1,...,x_n)$ is an input vector and $\mathbf{g}=(g_1,...,g_n)$ is a vector of continuous weighting functions.

Obviously, generalized mixture operators are continuous and idempotent. A generalized mixture operator based on the ordinal approach can be defined as follows.

Definition 4 An ordered generalized mixture operator $M'_{\mathrm{g}}: \quad [0,1]^n \to [0,1]$ is given by

$$M'_{\mathrm{g}}(x_1,...,x_n)=\frac{\sum_{i=1}^{n} g_i\left(x_{(i)}\right)\cdot x_{(i)}}{\sum_{i=1}^{n} g_i\left(x_{(i)}\right)}, \tag{16}$$

where $\mathbf{g}=(g_1,...,g_n)$ is a vector of continuous weighting functions and $\left(x_{(1)},...,x_{(n)}\right)$ is a non-decreasing permutation of an input vector.

An ordered generalized mixture operator is a generalization of an OWA operator [19], corresponding to constant weighting functions $g_i = w_i$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$.

However, a mixture operator need not be non-decreasing. Marques-Pereira and Pasi [6] stated the the first sufficient condition for a weighting function g in order to a mixture operator (8) is to be non-decreasing. It can be written as follows:

Proposition 1 Let $g:[0,1]\to \left]0,\infty\right[$ be a non-decreasing smooth weighting function which satisfies the next condition:

$$0 \le g'(x) \le g(x) \tag{17}$$

for all $x \in [0,1]$. Then $M_g : [0,1]^n \to [0,1]$ is an aggregation operator for each $n \in N$, $n > 1$.

We have generalized sufficient condition (17) in our previous work. In the next part we recall more general sufficient conditions mentioned in [7], [14-16].

From (14) we see that

$$\frac{\partial M_g}{\partial x_1} = \frac{\left(g(x_1) + g'(x_1)\cdot x_1\right)\cdot \sum_{i=1}^{n} g(x_1) - \left(\sum_{i=1}^{n} g(x_1)\cdot x_1\right)\cdot g'(x_1)}{\left(\sum_{i=1}^{n} g(x_1)\right)^2} \geq 0 \tag{18}$$

if and only if

$$g^2(x_1) + \alpha\left(g(x_1) + g'(x_1)\cdot(x_1 - \beta)\right) \geq 0, \tag{19}$$

where

$\alpha = \sum_{i=2}^{n} g(x_i)$ and $\alpha \cdot \beta = \sum_{i=2}^{n} g(x_i)\cdot x_i$, and thus necessarily $\beta \in [0,1]$ and $\alpha \in [(n-1)\cdot g(0), (n-1)\cdot g(1)]$.

Now it is easy to see that (17) implies (19). However, (19) is satisfied also whenever

$$g(x_1) + g'(x_1)\cdot(x_1 - \beta) \geq 0 \tag{20}$$

for each $x_1 \in [0,1]$ and each $\beta_1 \in [0,1]$.

Because $g'(x_1) \geq 0$, (20) is fulfiled whenever

$$0 \leq g'(x)\cdot(1-x) \leq g(x) \text{ for all } x \in [0,1].$$

We have just shown a sufficient condition more general than (17).

Proposition 2 Let $g : [0,1] \to]0,\infty[$ be a non-decreasing smooth weighting function which satisfies the condition:

$$0 \leq g'(x)\cdot(1-x) \leq g(x) \tag{21}$$

for all $x \in [0,1]$. Then $M_g : [0,1]^n \to [0,1]$ is an aggregation operator for each $n \in N$, $n > 1$.

Moreover, we have improved sufficient condition (21), but constrained by n.

Proposition 3 For a fixed $n \in N$, $n > 1$, let $g : [0,1] \to]0,\infty[$ be a non-decreasing smooth weighting function satisfying the condition:

$$\frac{g^2(x)}{(n-1)\cdot g(1)} + g(x) \geq g'(x)\cdot(1-x) \tag{22}$$

for all $x \in [0,1]$.Then $M_g : [0,1]^n \to [0,1]$ is an aggregation operator. In the next proposition we introduce a sufficient condition for the non-decreasing-ness of generalized mixture operators.

Proof. Minimal value of $g(x_1) + g'(x_1) \cdot (x_1 - \beta)$ for $\beta \in [0,1]$ is attained for $\beta = 1$, i. e., it is $g(x_1) + g'(x_1) \cdot (x_1 - 1)$. Therefore, (19) is surely satisfied whenever

$$\frac{g^2(x_1)}{\alpha} + g(x_1) \geq g'(x_1) \cdot (1 - x_1).$$

Suppose that (22) holds. Then

$$\frac{g^2(x_1)}{\alpha} + g(x_1) \geq \frac{g^2(x_1)}{(n-1) \cdot g(1)} \geq g'(x_1) \cdot (1 - x_1),$$

i. e., (19) is satisfied and thus g is a fitting weighting function.

□

In the next proposition we introduce a sufficient condition for the non-decreasing-ness of generalized mixture operators.

Proposition 4 For a fixed $n \in N$, $n > 1$, $i = 1,2,...,n$, let $g_i : [0,1] \to]0, \infty[$ be a non-decreasing smooth weighting functions, such that

$$\frac{g_i^2(x)}{\sum_{j \neq i} g_j(1)} + g_i(x) \geq g_i'(x) \cdot (1 - x) \tag{23}$$

for all $x \in [0,1]$. Then $M_{\mathrm{g}}' : [0,1]^n \to [0,1]$, where $\mathbf{g} = (g_1, ..., g_n)$, is an aggregation operator.

4 Maximal Premium Determined by a Personal Utility Function

In practice, the utility function can be determined empirically by a personal interview made by a decision maker. In our opinion, there are at least two suitable ways to do this. The first one is based on an interview which provides us with probabilities estimated by an interviewed subject; the second one on a game with known probabilities where the interviewed subject gives us only information about a personal breaking point. The personal breaking point is the amount of wealth at which our individual is changed from risk averse to a risk seeker, or vice versa. An appropriate curve for a risk averse and risk loving part is then selected from the theoretical utility functions.

4.1 A Personal Utility Function – a Probability-oriented Approach

Following this approach a personal utility function can easily be constructed from the information gleaned from a short interview using the classical regression analysis. The decision maker can use this function in any personal decision analysis in which the payoff falls between 0 and 30000 €. Now we recall the interview, which is compiled as follows [4].

Let us suppose you are owner of an investment which brings you zero payoff now or a loss of 30000 €. However, you have a possibility to step aside from this investment under the penalty in the amount of a sequence: A: 1000 €, B: 5000 €, C: 10000 €, D: 15000 €, E: 25000 €. Your portfolio manager can provide you with information expressing the probability loosing the 30000 €. Think. What would be the biggest probability of the loss, so that you retain the above mentioned investment? Only a few well-proportioned graphic points are required. From our interview we took the respective person's data points $(0,1)$, $(-1000,0.8)$, $(-5000,0.75)$, $(-10000,0.60)$, $(-15000,0,60)$, $(-25000,0.40)$, $(-30000,0.00)$, and created the appropriate utility function of our respondent as shown in Fig. 3. This curve has an interesting shape that reflects our respondent's approach to risk. The different personal utility functions for our respondent were created using the IBM SPSS 18.0 system for the purpose of comparison. The maximal premium $P^{\max}$ was calculated by u^{-1} inverse function to the utility equilibrium equation (4)

$$P^{\max} = w - u^{-1}\left(E\left[u(w - X)\right]\right) \tag{24}$$

with system Mathematica 5.

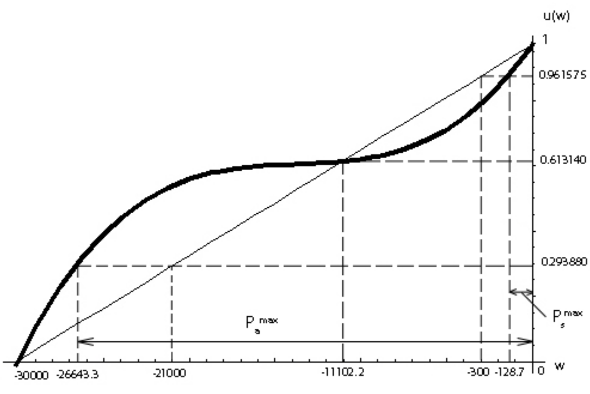

Figure 3
A utility function and the expected utility of our respondent (the function 2 from Table 1)

Utility functions are used to compare investments mutually. For this reason, we can scale a utility function by multiplying it by any positive constant and (or) transfer it by adding any other constant (positive or negative). This kind of transformation is called a positive affine transformation. All our results are the same with respect to such a transformation. Quadratic and cubic utility functions are written in Table 1. On the basis of statistical parameters (adjusted R square, p-values) we can assume that the cubic function is the best fitting function. Moreover, Table 1 also consists of appropriate expected utilities expressed by linear functions.

Remark 3 Expected utilities (for the utility functions from Table 1) can be calculated by means of a linear function which is assigned uniquely by points $(-30000, u(-30000))$ and $(0, u(0))$, or by the formula (8), alternatively. In both cases we get the same values for the expected utilities.

In Figure 3 you can see the personal utility function of our respondent, as well as three interesting points that are highlighted (also in Table 3). Maximal premium $P_a^{\max}$ represents the area where our respondent is risk averse, and $P_s^{\max}$, where he is risk seeking (loving).

Table 1
A utility function and the expected utility

	A utility function and the expected utility
1	$u(x) = -2.820 \cdot 10^{-10} x^2 + 1.856 \cdot 10^{-5} x + 0.904$ $E[u(x)] = 27.02 \cdot 10^{-6} x + 0.904$
2	$u(x) = 1.310 \cdot 10^{-13} x^3 + 5.383 \cdot 10^{-9} x^2 + 7.588 \cdot 10^{-5} x + 0.971$ $E[u(x)] = 3.23 \cdot 10^{-5} x + 0.971$

From Table 2 you can see that the insured person is willing to pay more than the expected loss to achieve his peace of mind".

Table 2
The expected utility and maximal premium with respect to a quadratic function

p **Probability**	$E[u]$ **with respect to quadratic function**	$P^{\max}$ **(€)**	$E[X]$ **(€)**	$P^{\min}$ **(€)**
0.00	0.904000	0.0	0.0	0
0.01	0.895894	433.9	300.0	301.69

0.05	0.863470	2115.7	1500.0	1508.10
0.10	0.822940	4110.7	3000.0	3015.34
0.20	0.741880	7808.5	6000.0	6027.24
0.30	0.660820	11197.3	9000.0	9035.69
0.40	0.579760	14336.3	12000.0	12040.70
0.50	0.498700	17293.4	15000.0	15042.40
0.60	0.417640	20079.0	18000.0	18040.60
0.70	0.336580	22725.4	21000.0	21035.50
0.80	0.255520	25251.4	24000.0	24027.00
0.90	0.174460	27672.4	27000.0	27015.20
1.00	0.093400	30000.0	30000.0	30000.00

Table 3

The expected utility and maximal premium with respect to a cubic function

p **Probability**	$E[u]$ **with respect to cubic function**	$P^{\max}$ **(€)**	$E[X]$ **(€)**	$P^{\min}$ **(€)**
0.00	0.971252	0.0	0.0	0,00
0.01	**0.961575**	**128.7**	**300.0**	301.69
0.05	0.922868	669.1	1500.0	1508.10
0.10	0.874485	1411.9	3000.0	3015.34
0.20	0.777717	3234.4	6000.0	6027.24
0.30	0.680950	6021.0	9000.0	9035.69
0.37	**0.613140**	**11102.2**	**11102.2**	**10986.70**
0.40	0.584182	18165.0	12000.0	12040.70
0.50	0.487415	22797.1	15000.0	15042.40
0.60	0.390647	25033.0	18000.0	18040.60
0.70	**0.293880**	**26643.3**	**21000.0**	21035.50
0.80	0.197112	27938.0	24000.0	24027.00
0.90	0.100345	29036.7	27000.0	27015.20
1.00	0.003577	30000.00	30000.0	30000.00

We determine the minimal premium by means of (7) with respect to the utility function for insurer $U(x) = \ln x$ with his basic capital $W = 2655513.51$ € and loss $X = 30000$ €.

The equation can be rewritten as follows:

$$U(W) = p \cdot U\left(W + P^{\min} - X\right) + (1-p) \cdot U\left(W + P^{\min}\right) \tag{25}$$

and hence

$$W = \left(W + P^{\min} - X\right)^{p} \cdot \left(W + P^{\min}\right)^{(1-p)}. \tag{26}$$

We determined individual minimal premiums with corresponding probability with the system Mathematica 5.

4.2 A Personal Utility Function– a Game-based Approach

Our expectation that our subject can appropriately estimate probabilities is the main drawback of the previous approach. In fact, we can doubt whether somebody without appropriate knowledge about probabilities can provide us with reliable answers. In order to avoid this problem we can assume a game with probabilities which are easy to understand, e.g. games based on coin tossing. Let us assume the following game. You have two possibilities: either to toss a coin with two possible results, head means you will get 10 €, tail means you will get nothing; or to choose 5 € without playing. What is amount of money for which you will start (stop) playing? It is easy to see that the expected value is the same in both cases and we make our decision about playing with respect to our personal utility function. The point at which we give up (stop) playing is the above mentioned breaking point. For simplicity we will assume the quadratic utility function. This approach allows us to combine easily personal utility functions to a group utility function using aggregation operators. The group utility function can represent a specific group of customers of our insurance company. Let us assume three utility functions based on different breaking, a utility function for $x = 29900$

$$u_1(x) = \begin{cases} -5.592778604 \cdot 10^{-10} \cdot (x-29900)^2 + 0.5 & \text{for } 0 \le x \le 29900 \\ 5 \cdot 10^{-5} \cdot (x-29900)^2 + 0.5 & \text{for } 29900 \le x \le 30000 \end{cases}, \tag{27}$$

a utility function for $x = 29800$

$$u_1(x) = \begin{cases} -5.63037701 \cdot 10^{-10} \cdot (x-29800)^2 + 0.5 & \text{for } 0 \le x \le 29800 \\ 1.25 \cdot 10^{-5} \cdot (x-29800)^2 + 0.5 & \text{for } 29800 \le x \le 30000 \end{cases}, \tag{28}$$

A utility function for $x = 29650$

$$u_1(x) = \begin{cases} -5.687489514 \cdot 10^{-10} \cdot (x-29650)^2 + 0.5 & \text{for } 0 \le x \le 29650 \\ 0.4081 \cdot 10^{-5} \cdot (x-29650)^2 + 0.5 & \text{for } 29650 \le x \le 30000 \end{cases}. \tag{29}$$

To construct the combined utility function we can use for example an ordered generalized mixture operator M'_{g} with weighting vector $\mathbf{g} = (g_1, g_2, g_3)$, where $g_1(x) = 0.2x + 0.8$, $g_2(x) = 0.5x + 0.5$ and $g_3(x) = 0.75x + 0.25$. Let us note that the selected weighting functions satisfy the conditions required for of non-decreasing aggregation operators.

Values 29900, 29800, 29650 we transform to the unit interval and aggregate them by means of M'_{g}. We obtain an aggregated value $M'_{\mathrm{g}} = 0.00575309$, and after transformation we have point of division $x = 29827.41$, where the insured is neutral to risk. On the basis of this division point we can create a new combined utility function

$$u(x) = \begin{cases} -5.620033657 \cdot 10^{-10} \cdot (x - 29827{,}41)^2 + 0.5 & \text{for } \ 0 \le x \le 29827{,}41 \\ 1.6785 \cdot 10^{-5} \cdot (x - 29827{,}41)^2 + 0.5 & \text{for } \ 29827{,}41 \le x \le 30000 \end{cases} \tag{30}$$

and appropriate expected utility

$$E(x) = \begin{cases} 1.6763 \cdot 10^{-5}\, x & \text{for } \ 0 \le x \le 29827{,}41 \\ 2.897039 \cdot 10^{-3} & \text{for } \ 29827{,}41 \le x \le 30000 \end{cases} \tag{31}$$

Table 4

The expected utility and maximal premium with respect to a function (28)

X	$w - X$	$E[u(w - X)]$	$w - P^{\max}$	$P^{\max}$	$P^{\min}$
0	30000,00	1,000000	30000,00	0,00	0
50	29950,00	0,855148	29972,90	27,10	50
100	29900,00	0,710296	29939,30	60,70	100
172,59	29827,41	0,500000	29827,40	172,60	172,59
200	29800,00	0,499541	28923,70	1076,30	200
300	29700,00	0,497864	27877,90	2122,10	300
1500	28500,00	0,477748	23535,00	6465,00	1500
3000	27000,00	0,452604	20644,10	9355,90	3000
6000	24000,00	0,402315	16643,50	13356,50	6000
9000	21000,00	0,352025	13600,90	16399,10	9000
12000	18000,00	0,301736	11044,90	18955,10	12000
15000	15000,00	0,251447	8797,37	21202,63	15000
18000	12000,00	0,201157	6767,79	23232,21	18000
21000	9000,00	0,150868	4902,97	25097,03	21000
24000	6000,00	0,100579	3168,28	26831,72	24000
27000	3000,00	0,050289	1539,73	28460,27	27000
30000	0,00	0,000000	0,00	30000,00	30000

The minimal premium we evaluated on the basis of formula (7) with the utility function for insurer $U(x) = \ln(x+1)$ with the system Mathematica. From Table 4 and also from the formula (7) you can see that the minimal premium is given by the size of the expected loss. A newly-gained utility function would be required for evaluating a decision with more extreme payoffs or if our respondent's attitudes change because of a new job or lifestyle change. Moreover, the utility function must be revised from the viewpoint of time.

Conclusions

We have shown two approaches to creating a personal utility function and we have calculated the maximum premium against the loss of 30000 € with respect to it. We think that the personal utility function of an insured person would be very important for an insurer. On the basis of the personal utility function the insurer would know what approaches to risk the customers have and thus, how they will behave towards their own wealth. Creating a utility function for the insurer is very difficult. Moreover, in our next work we want to investigate the insurer's utility function and we want to determine the minimal premium against the loss of 30000 € with respect to a concrete real insurer's utility function.

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