Cold duplication and survival equivalence in the case of gamma – Weibull distributed composite systems

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Abstract: *The reliability of composite system (series, parallel) is improved by* **(i)** *reduction method, and by* **(ii)** *cold duplication, considering the system's survival functions. Then these reliability improvement methods are compared mentioning that the hot duplication method was studied recently by* Pogány *et al.* [8].

In this companion article to [8]*, related survival equivalence functions and pointwise survival equivalence factors are derived in all cases when the components lifetime distribution follow the gamma–Weibull distribution introduced recently by* Leipnik *and* Pearce*, and studied intensively by* Nadarajah *and* Kotz*, then by* Pogány *and* Saxena*.*

Keywords: *Cold–duplication method, composite system, Fox–Wright generalized hypergeometric function, gamma–Weibull distribution, parallel system, reduction method, reliability equivalence factor, reliability function, series system, Srivastava–Daoust generalized Kampé de Fériet hypergeometric function, survival equivalence function, survival function.*

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1 Introduction

The *reliability equivalence* has been introduced by Råde [9], who developed this concept to improve the reliability of various systems [10]. Following Sarhan [11] and Xia and Zhang [18], the reliability equivalence factor (**REF**) is a *factor by which the failure rates of some of the system's components should be reduced in order to reach equality of the reliability of another better system.*

Detailed account and numerous unification of Råde's ideas can be found in Sarhan's articles [11, 12, 13, 14]. He studied among others the reliability of composite, i.i.d. series/parallel systems decreasing their failure rates, and using hot-, and cold–duplication. Also, he considered parameter estimation in composite systems and related questions (mainly when the life distribution of components is exponential) as well, consult Sarhan's cited articles and the references therein.

Råde discussed three different methods to improve the systems reliability: **1.** Improving the quality of $r \leq n$ components by decreasing their hazard rates; **2.** adding a hot component to the system, and **3.** adding a cold (redundant) component to the system [9, 10].

Following Råde's traces Sarhan [12] has introduced more general methods in systems reliability improvement either modifying the method **1.** by introducing a reduction coefficient $\rho \in (0,1)$; or completing the system by cold redundant standby components connected with some components by perfect random switches. Both authors considered components having exponential life distributions.

Finally, we point out some recent exceptions, e.g. the work by Xia and Zhang [18], where the improvement of the reliability of the parallel system of gamma–distributed components is considered in both hot–, and cold–duplication manner, and the very recent article [8] by Pogány *et al.* in which the authors show that the reduction method is actually not punctually superior to the classical Hot duplication method in series and parallel composite systems which components are gamma–Weibull distributed.

The hazard rate is constant only for exponential life distribution; the gamma–distribution has a functional hazard rate. So, Sarhan's results concerning the in parallel connected systems having exponential distribution are generalized in [18] taking instead of exponential distribution its generalization such as the gamma–distribution. In the same time Xia and Zhang unify *mutatis mutandis* the concept of REF.

At this point we introduce a new concept, reads as follows: *The survival equivalence function* (**SEF**) *is a function by which the survival function of the considered system has to be multiplied in order to reach pointwise equality of the survival function of another better system.*

In this article we obtain the SEF in general case, when each component's life distribution is described by a r.v. ξ having distribution function $F_\theta(x)$. The systems are distinguished by their components connection topology: **(i)** (S) with independent identical components (**i.i.c.**) in series connected, and **(ii)** (P) which components are connected in parallel.

Composite system SEFs are obtained when the systems consist from i.i.c. possessing gamma–Weibull $gW(\theta)$ life–distribution which has been intensively studied by Leipnik and Pearce [4], Nadarajah and Kotz [6] and Pogány and Saxena [7]. Parameter estimation in Weibull models can be seen in [2].

Since the case of Hot Duplication was already discussed in detail in [8], we concentrate to the Cold Duplication case, comparing them by the reduction method applied simultaneously to the same size composite system having identical topology.

2 Survival functions of composite systems

In this section of introductionary character following mainly the notations used in [8], we recall in short the basic probabilistic notations, concepts and tools we will need frequently in the sequel.

Let ξ be a random variable defined over a standard probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with the cumulative distribution function (**CDF**) $F_\theta(x) = \mathbb{P}\{\xi < x\}, x \in \mathbb{R}$ where θ stands for the parameter vector. The related *reliability function*

$$R_\theta(x) = \mathbb{P}\{\xi \ge x\} = 1 - F_\theta(x) \qquad \big(x \in \mathbb{R}\big)\,.$$

Let us consider for the sake of simplicity a system (S) consisting of n i.i.c. connected in series. The lifetime of any component is assumed to be a r.v. ξ having the cumulative distribution function $F_\theta(x)$. Taking n independent replicæ of ξ, the *survival function*, i.e. the common reliability function R_S of the composite system becomes

$$\mathscr{S}_{\theta,\mathsf{S}}(x) = \big[R_\theta(x)\big]^n = \big[1 - F_\theta(x)\big]^n\,. \tag{1}$$

However, in the case of parallel system (P) of n i.i.c. the related survival function

$$\mathscr{S}_{\theta,\mathsf{P}}(x) = 1 - \big[1 - R_\theta(x)\big]^n = 1 - \big[F_\theta(x)\big]^n\,. \tag{2}$$

Hence, it can be easily seen that we can express the both survival functions $R_{\theta,B}(x)$, $B \in \{\mathsf{S},\mathsf{P}\}$ either in therms of the reliability function R_θ of a consisting component, or in therms of the probability distribution function F_θ, and of the system's components number n. Finally, we point out that a reliability function is of bounded variation $(R_\theta(-\infty) = 1, R_\theta(\infty) = 0)$, monotone non-increasing and left–continuous. Any such function R_θ possesses a *generalized inverse*

$$\mathscr{R}^\star(y) := \inf\{x\colon R_\theta(x) \ge y\} \qquad \big(0 \le y < 1\big)\,.$$

More precisely, if R_θ is strong monotone, then $\mathscr{R}^\star \equiv R_\theta^{-1}$ in the usual sense. Recall, that in reliability theory it is convenient to consider r.v. ξ such that $R_\theta(x) \le 1$ only for $x > 0$, equivalently $\operatorname{supp}\big(F_\theta\big) = \operatorname{supp}\big(1 - R_\theta\big) = \mathbb{R}_+$ [1].

Finally, let as denote the SEF by $\mathbf{r}(x)$. According to the defintion of SEF, it will be

$$\mathbf{r}_B^D(x)\,\mathscr{S}_{\theta,B}(x) = \mathscr{S}_{\theta,B}^D(x) \qquad \big(B \in \{\mathsf{S},\mathsf{P}\}\big)\,,$$

where $\mathscr{S}_{\theta,B}^D(x)$ denotes the survival function of a better, more reliable system, where the superscript D will be fixed later.

[1] The support of some g coincides with the set $\operatorname{supp}(g) = \overline{\{x\colon g(x)\}}$, where bar means closure.

2.1 Reduction method

Let us consider the systems $(\mathsf{S}), (\mathsf{P})$ such that become $(\mathsf{S}_r), (\mathsf{P}_r)$ by improving r, $1 \leq r \leq n$ of its components assuming that their reliability is enlarged using

$$R_\theta(\rho x) \qquad \big(\rho \in (0,1)\big)$$

instead of the original reliability $R_\theta(x)$ [2]. The associated survival functions are:

$$\begin{aligned} \mathscr{S}^{\rho}_{\theta,\mathsf{S}_r}(x) &= \big[R_\theta(\rho x)\big]^r \big[R_\theta(x)\big]^{n-r}, \\ \mathscr{S}^{\rho}_{\theta,\mathsf{P}_r}(x) &= 1 - \big[1 - R_\theta(\rho x)\big]^r \big[1 - R_\theta(x)\big]^{n-r}. \end{aligned}$$

The multiplication of the original survival functions (1), (2) by SEF reduces r arguments in the product to $\rho x, \rho \in (0,1)$. By definition of the SEF we have

$$\begin{aligned} \mathbf{r}^{\rho}_{\mathsf{S}_r}(x) &= \left[\frac{R_\theta(\rho x)}{R_\theta(x)}\right]^r, \\ \mathbf{r}^{\rho}_{\mathsf{P}_r}(x) &= \frac{1 - [1 - R_\theta(\rho x)]^r [1 - R_\theta(x)]^{n-r}}{1 - [1 - R_\theta(x)]^n}. \end{aligned}$$

Choosing some convenient $\rho \in (0,1)$, $r \in \{1, \cdots, n\}$ we can work with exact SEF functions such that are associated with the reduction method.

2.2 Cold duplication method

We improve p, $1 \leq p \leq n$ components by cold duplication method, i.e. *p standby* components are connected in parallel by an identical one with a perfect switch getting (S^C_p), (P^C_p). We can express the survival function $R^{(1)}_\theta(x)$ of the connected *working* $\leftrightarrow$ *standby* components pair by the autoconvolution of $R_\theta(x)$, i.e.

$$R^{(1)}_\theta(x) = 1 - F_\theta * F_\theta(x) = 1 - \int_{\mathbb{R}} F_\theta(x-t) \mathrm{d}F_\theta(t) = - \int_0^x R_\theta(x-t) \mathrm{d}R_\theta(t),$$

being $R_\theta(x) = 0$ for negative values of the argument, where $*$ denotes the convolution operator to CDFs, *id est* to reliability functions as well. The corresponding survival functions become

$$\begin{aligned} \mathscr{S}^{C}_{\theta,\mathsf{S}_p}(x) &= \big[R^{(1)}_\theta(x)\big]^p \big[R_\theta(x)\big]^{n-p} \\ &= \left(-\int_0^x R_\theta(x-t)\mathrm{d}R_\theta(t)\right)^p \big[R_\theta(x)\big]^{n-p}, \\ \mathscr{S}^{C}_{\theta,\mathsf{P}_p}(x) &= 1 - \big[R^{(1)}_\theta(x)\big]^p \big[1 - R_\theta(x)\big]^{n-p} \\ &= 1 - \left(-\int_0^x R_\theta(x-t)\mathrm{d}R_\theta(t)\right)^p \big[1 - R_\theta(x)\big]^{n-p}. \end{aligned}$$

[2] Being $R_\theta \downarrow$ monotone nonincreasing, the concept needs only a new technological support, the introduced mathematical model is indeed well defined.

Now, we equalize the reliabilities of the improved system, obtained by the reduction method from one, and the cold duplication method from the other hand. Thus

$$\mathscr{S}^{\rho}_{\theta,\mathsf{S}_r}(x) = \mathscr{S}^{C}_{\theta,\mathsf{S}_p}(x)\,;$$

this equation reduces to

$$R_\theta(\rho x) = \big[R_\theta(x)\big]^{1-p/r}\left(-\int_0^x R_\theta(x-t)\mathrm{d}R_\theta(t)\right)^{p/r}. \tag{3}$$

This equation possesses solution only when the right–hand expression in (3) is less then 1. Since $R_\theta(\cdot) \le 1$, we conclude

$$\begin{aligned}-\int_0^x R_\theta(x-t)\mathrm{d}R_\theta(t) &= \int_0^x R_\theta(x-t)\mathrm{d}F_\theta(t)\\ &\le \int_0^x \mathrm{d}F_\theta(x) = F_\theta(x) = 1-R_\theta(x)\,,\end{aligned}$$

being F_θ left–continuous. Now, looking for the maximum of the function $\widetilde{g}(x) = x^{1-p/r}(1-x)$, $x \in (0,1)$ we have

$$\widetilde{g}(p/r) := \max_{0<x<1}\widetilde{g}(x) = g\Big(\frac{r-p}{2r-p}\Big) = \frac{(1-p/r)^{1-p/r}}{(2-p/r)^{2-p/r}} \qquad \big(p \le r\big)\,.$$

Obviously $\widetilde{g}(p/r) \le 1$, so does *a fortiori* the right–side expression in the display (3). But this means $p \le r$. Hence, we have to incorporate the condition $p \le r$ throughout.

Finally, we get the pointwise survival equivalence factor related to S_r:

$$\rho^C_{\mathsf{S}} = x^{-1}\mathscr{R}^\star\left[\big[R_\theta(x)\big]^{1-p/r}\left(-\int_0^x R_\theta(x-t)\mathrm{d}R_\theta(t)\right)^{p/r}\right]. \tag{4}$$

Solving now the equation $\mathscr{S}^{\rho}_{\theta,\mathsf{P}_r}(x) = \mathscr{S}^{C}_{\theta,\mathsf{P}_p}(x)$ by a similar procedure we arrive at

$$\rho^C_{\mathsf{P}} = x^{-1}\mathscr{R}^\star\left[1-\big[1-R_\theta(x)\big]^{1-p/r}\left(-\int_0^x R_\theta(x-t)\mathrm{d}R_\theta(t)\right)^{p/r}\right], \tag{5}$$

which presents the pointwise factor related to parallel system P_r.

Theorem 1. *The pointwise cold–duplication* SEF *associated to n i.i.c. series composite system* (S^C_p), *$p \le r$ is given by*

$$\mathbf{r}^C_{\mathsf{S}_r}(x) = \left[\frac{R_\theta(\rho^C_{\mathsf{S}}x)}{R_\theta(x)}\right]^r.$$

The factor ρ^C_{S} is presented in the display (4).

The pointwise cold–duplication SEF *corresponding to parallel system* (P^C_p) *is*

$$\mathbf{r}^C_{\mathsf{P}_r}(x) = \frac{1-[1-R_\theta(\rho^C_{\mathsf{P}}x)]^r[1-R_\theta(x)]^{n-r}}{1-[1-R_\theta(x)]^n}$$

where ρ^C_{P} one can express by (5) *and p/r is unrestricted.*

Corollary 1.1. *Let the distribution function F_θ be strong monotone. Then*

$$\mathbf{r}^C_{\mathsf{S}_r}(x) = \big[R_\theta(x)\big]^{-p}\left(-\int_0^x R_\theta(x-t)\mathrm{d}R_\theta(t)\right)^p \qquad \big(p\le r\big),$$

$$\mathbf{r}^C_{\mathsf{P}_r}(x) = \frac{1-[1-R_\theta(x)]^{n-p}\Big(-\int_0^x R_\theta(x-t)\mathrm{d}R_\theta(t)\Big)^p}{1-[1-R_\theta(x)]^n}\,.$$

The proof is based again on the existence of inverse R_θ^{-1} since the reliability function $R_\theta = 1-F_\theta$ is monotone by assumption.

3 gamma–Weibull distribution and related reliability functions

Leipnik and Pearce [4] introduced recently a new distribution referred to as the *gamma–Weibull distribution* (gW); in fact, they renormalize the multiplied densities of the gamma–, and the Weibull–distributions to give a new density function. Nadarajah and Kotz [6] pointed out that it is enough to take four parameters to define the $gW(\theta)$ distribution having probability density function (**PDF**)

$$f_{gW}(x) = Kx^{\alpha-1}\exp\big\{-\mu x - ax^\kappa\big\}\,\chi_{(0,\infty)}(x) \qquad \big(\theta := (\alpha,\mu,a,\kappa)>0\big), \tag{6}$$

where $\chi_A(x)$ denotes the characteristic function of the set A, i.e. $\chi_A(x)=1, x\in A$ and $\chi_A(x)=0, x\notin A$. So, in this case the r.v. ξ is said to have $gW(\theta)$ distribution, such that we write $\xi\sim gW(\theta)$.

Before we characterize the $gW(\theta)$–distribution, we introduce certain notations and results we need for the further exposition.

Here, and in what follows, ${}_p\Psi_q$ denotes the Fox–Wright generalization of the hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters, defined by

$$\begin{aligned}{}_p\Psi_q\left[\begin{array}{c}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),\cdots,(b_q,\beta_q)\end{array}\bigg|\, x\right] &= {}_p\Psi_q\left[\begin{array}{c}(a_p,\alpha_p)\\ \\ (b_q,\beta_q)\end{array}\bigg|\, x\right]\\ &:= \sum_{m=0}^\infty \frac{\prod_{\ell=1}^p\Gamma\big(a_\ell+\alpha_\ell m\big)}{\prod_{\ell=1}^q\Gamma\big(b_\ell+\beta_\ell m\big)}\,\frac{x^m}{m!}\end{aligned} \tag{7}$$

under the parameter constraint

$$\alpha_\ell\in\mathbb{R}_+,\, \ell=\overline{1,p};\quad \beta_j\in\mathbb{R}_+,\, j=\overline{1,q};\quad 1+\sum_{\ell=1}^q\beta_\ell-\sum_{j=1}^p\alpha_j>0 \tag{8}$$

for suitably bounded values of $|x|$ in terms of Euler's gamma function:

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt \qquad \big(\Re\{s\}>0\big).$$

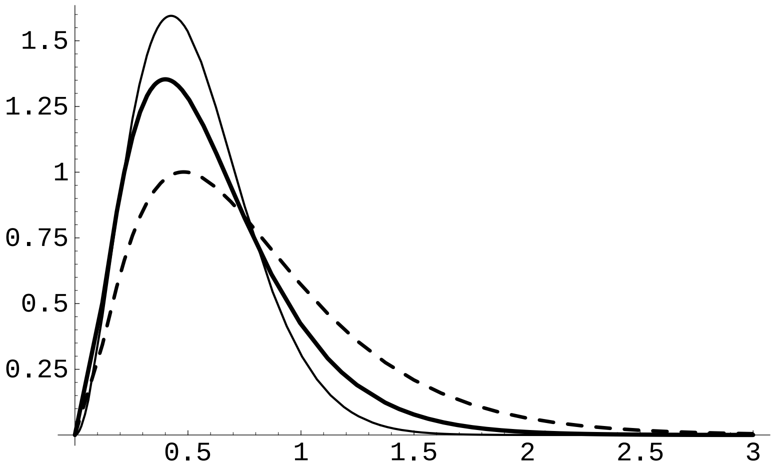

Figure 1
gamma–Weibull density functions $f_{gW}(x)$ with $\alpha = 3$, $\mu = 3$, $a = 2$; $\kappa = 0.363$ dashed line, $\kappa = 1$ solid line and $\kappa = 2$ thin solid line.

We note that in (7) the empty product means unity see e.g. [17]. The *upper incomplete gamma–function* [3, **8.350** 2.] reads as follows:

$$\Gamma(s,z) := \int_z^\infty t^{s-1} e^{-t}\, dt\,; \qquad \lim_{z\to 0} \Gamma(s,z) = \Gamma(s)\,.$$

Replacing in series expansion (7) of ${}_p\Psi_q$ all gamma–function terms by upper incomplete gamma–function terms having identical second variables, we get the *upper incomplete Fox–Wright Psi–Function* firstly considered by Srivastava and the first author [16]:

$$\begin{aligned} {}_p\Psi_q\left[\begin{array}{c} (a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ \\ (b_1,\beta_1),\cdots,(b_q,\beta_q) \end{array}\middle|^{\Gamma} (x,z)\right] &= {}_p\Psi_q\left[\begin{array}{c} (a_p,\alpha_p) \\ \\ (b_q,\beta_q) \end{array}\middle|^{\Gamma} (x,z)\right] \\ &= \sum_{m=0}^\infty \frac{\prod_{\ell=1}^p \Gamma\big(a_\ell + \alpha_\ell m, z\big)}{\prod_{\ell=1}^q \Gamma\big(b_\ell + \beta_\ell m, z\big)} \frac{x^m}{m!} \qquad \big(z \ge 0\big) \end{aligned}$$

for all parameters such that satisfy (8), that is for the parameter space

$$\alpha_k \in \mathbb{R}_+,\, k = \overline{1,p}; \qquad \beta_j \in \mathbb{R}_+,\, j = \overline{1,q}; \qquad 1 + \sum_{k=1}^q \beta_\ell - \sum_{j=1}^p \alpha_j > 0\,.$$

Subsequently, the normalizing constant $K = K(\theta)$ of the probability density function (6) is [7, Eq. (9)]

$$K^{-1} = K^{-1}(\theta) = \begin{cases} \mu^{-\alpha}\,{}_1\Psi_0\left[\begin{array}{c} (\alpha,\kappa) \\ \hline \end{array}\middle| -\dfrac{a}{\mu^\kappa}\right] & 0<\kappa<1 \\[2ex] \dfrac{\Gamma(\alpha)}{(\mu+a)^\alpha} & \kappa = 1 \\[2ex] \dfrac{1}{\kappa a^{\alpha/\kappa}}\,{}_1\Psi_0\left[\begin{array}{c} (\alpha/\kappa,1/\kappa) \\ \hline \end{array}\middle| -\dfrac{\mu}{a^{1/\kappa}}\right] & \kappa>1 \end{cases}, \qquad (9)$$

where ${}_1\Psi_0[\cdot]$ stands for the so–called *confluent complete Fox–Wright generalization of the hypergeometric function*, introduced *via* the series (7) above.

We only remark that in the case $\kappa > 1$ the reciprocal of the constant

$$K^{-1} = K^{-1}(\theta) = \int_0^\infty x^{\alpha-1}\exp\big\{-\mu x - ax^\kappa\big\}\,\mathrm{d}x$$

is obtainable by expanding $e^{-\mu x}$ into Maclaurin series, integrating termwise, then summing up the resulting expression, while the case $0<\kappa<1$ we handle expanding the term e^{-ax^κ}.

Finally, the remaining case $\kappa = 1$ coincides with the two–parameter gamma–distribution; the approaching parameters are α and $\beta = \mu + a$, [7].

Now, the reliability function $R_{gW}(x)$ of one–component, having $gW(\theta)$ life distribution, related to the probability distribution function $f_{gW}(x)$ becomes

$$R_{gW}(x) = \chi_{(0,\infty)}(x)\begin{cases} \dfrac{{}_1\psi_0\left[\begin{array}{c} (\alpha,\kappa) \\ \hline \end{array}\middle|^{\Gamma} \big(-a\mu^{-\kappa},\mu x\big)\right]}{{}_1\Psi_0\left[\begin{array}{c} (\alpha,\kappa) \\ \hline \end{array}\middle| -\dfrac{a}{\mu^\kappa}\right]} & 0<\kappa<1 \\[3ex] \dfrac{\Gamma\big(\alpha,(\mu+a)x\big)}{\Gamma(\alpha)} & \kappa=1 \\[3ex] \dfrac{{}_1\psi_0\left[\begin{array}{c} (\alpha/\kappa,1/\kappa) \\ \hline \end{array}\middle|^{\Gamma} \big(-\mu a^{-1/\kappa},ax^\kappa\big)\right]}{{}_1\Psi_0\left[\begin{array}{c} (\alpha/\kappa,1/\kappa) \\ \hline \end{array}\middle| -\dfrac{\mu}{a^{1/\kappa}}\right]} & \kappa>1 \end{cases} \qquad (10)$$

where ${}_1\psi_0$ denotes the *confluent upper incomplete Fox–Wright Psi–function*, while for $x \le 0, R_\theta(x) \equiv 1$. Since $x > 0$ is understood, we omit to write $\chi_{(0,\infty)}(x)$ in the sequel.

In order to prove (10), let us assume $\kappa > 1$. Than we have

$$\begin{aligned} R_{gW}(x) &= K\int_x^\infty t^{\alpha-1}e^{-\mu t - at^\kappa}\mathrm{d}t \\ &= K\sum_{n=0}^\infty \frac{(-\mu)^n}{n!}\int_x^\infty t^{\alpha+n-1}\exp\big\{-at^\kappa\big\}\mathrm{d}t \\ &= \frac{K}{\kappa a^{\alpha/\kappa}}\sum_{n=0}^\infty \frac{\big(-\mu/a^{1/\kappa}\big)^n}{n!}\int_{ax^\kappa}^\infty y^{(\alpha+n)/\kappa-1}e^{-y}\mathrm{d}y \\ &= \frac{K}{\kappa a^{\alpha/\kappa}}\sum_{n=0}^\infty \frac{\Gamma\big((\alpha+n)/\kappa, ax^\kappa\big)}{n!}\Big(-\frac{\mu}{a^{1/\kappa}}\Big)^n, \end{aligned}$$

such that guarantees (10). In the case $\kappa \in (0,1)$ we repeat the earlier procedure in getting (9). The case $\kappa = 1$ is obvious.

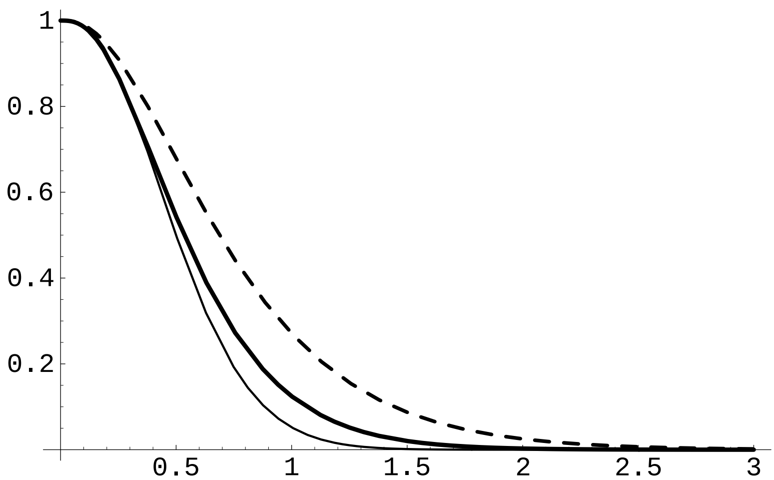

Figure 2
gamma–Weibull reliability functions $R_{gW}(x)$ with $\alpha = 3$, $\mu = 3$, $a = 2$; $\kappa = 0.363$ dashed line, $\kappa = 1$ solid line and $\kappa = 2$ thin solid line.

Theorem 2. *Let us consider* $(\mathsf{S}),(\mathsf{P})$ *consisting from n i.i.c. such that have* $gW(\theta)$ *life–distributions. Then the related survival functions have the form*

$$\begin{aligned} \mathscr{S}_{gW,\mathsf{S}}(x) &= \big[R_{gW}(x)\big]^n \\ \mathscr{S}_{gW,\mathsf{P}}(x) &= 1-\big[1-R_{gW}(x)\big]^n, \end{aligned}$$

where $R_{gW}(x)$ *is displayed in* (10).

Proof. By (1), (2) we build easily the survival functions of systems $(\mathsf{S}),(\mathsf{P})$ applying n i.i.d. replicæ of a r.v. $\xi \sim gW(\theta)$ such that describes the life–distribution of all involved components. □

Remark 1. The pointwise SEF $\mathbf{r}^H$ and survival equivalence factors ρ^H for both -series and parallel composite systems are already given by Pogány *et al.* in [8]. □

Finally, it remains to expose the results upon the pointwise SEF $\mathbf{r}^C$ and survival equivalence factor ρ^C for the series and parallel composite systems, assuming $1 \leq p \leq n$ components are improved by cold–duplication method. At this moment the need of new mathematical tool arises. Let us introduce the Srivastava–Daoust generalized Kampé de Fériet hypergeometric function in two variables [15, Eq. (2.1)]:

$$S^{A:B;B'}_{C:D;D'}\left(\begin{array}{c} \big[(a)\colon \theta,\phi\big]\colon \big[(b)\colon \psi\big]\,;\big[(b')\colon \psi'\big] \\ \\ \big[(c)\colon \delta,\varepsilon\big]\colon \big[(d)\colon \eta\big]\,;\big[(d')\colon \eta'\big] \end{array}\middle|\, \begin{array}{c} x \\ y \end{array}\right)$$
$$= \sum_{m,n=0}^{\infty} \frac{\prod\limits_{j=1}^{A}\Gamma(a_j+m\theta_j+n\phi_j)\prod\limits_{j=1}^{B}\Gamma(b_j+m\psi_j)\prod\limits_{j=1}^{B'}\Gamma(b'_j+n\psi'_j)}{\prod\limits_{j=1}^{C}\Gamma(c_j+m\delta_j+n\varepsilon_j)\prod\limits_{j=1}^{D}\Gamma(d_j+m\eta_j)\prod\limits_{j=1}^{D'}\Gamma(d'_j+n\eta'_j)}\,\frac{x^m y^n}{m!\,n!} \tag{11}$$

where the coefficients $\theta_1,\phi_1,\psi_1,\psi'_1,\delta_1,\varepsilon_1,\eta_1,\eta'_1,\cdots,\theta_A,\phi_A,\psi_B,\psi'_{B'},\delta_C,\varepsilon_C,\eta_D,\eta'_{D'}$ are real and positive and (a) denotes a sequence of A parameters $a_1,\cdots,a_A$. The convergence of (11) is ensured for

$$1+\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\eta_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j>0$$
$$1+\sum_{j=1}^{C}\varepsilon_j+\sum_{j=1}^{D}\eta'_j-\sum_{j=1}^{A}\phi_j-\sum_{j=1}^{B}\psi'_j>0\,.$$

In the cold duplication method the reliability of two in parallel connected identical components has to be determined when one of them is active and the other one is standby. Assume that both of them possess $gW(\theta)$ life–distribution. We calculate the related PDF $\varphi(x)$ using the autoconvolution of the input $gW(\theta)$ density (6). So, we have

$$\begin{aligned}
\varphi(x) &= \int_{\mathbb{R}} f(x-t)f(t)\mathrm{d}t = K^2 e^{-\mu x}\int_0^x \big[t(x-t)\big]^{\alpha-1} e^{-a[(x-t)^\kappa+t^\kappa]}\,\mathrm{d}t \\
&= K^2 e^{-\mu x}x^{2\alpha-1}\int_0^1 \big[t(1-t)\big]^{2\alpha-1} e^{-ax^\kappa[(t^\kappa+(1-t)^\kappa]}\mathrm{d}t \\
&= K^2 e^{-\mu x}x^{2\alpha-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-ax^\kappa)^{m+n}}{m!\,n!}\int_0^1 t^{\kappa m+\alpha-1}(1-t)^{\kappa n+\alpha-1}\mathrm{d}t \\
&= K^2 e^{-\mu x}x^{2\alpha-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\kappa m+\alpha)\Gamma(\kappa n+\alpha)}{\Gamma\big(\kappa(m+n)+2\alpha\big)}\,\frac{(-ax^\kappa)^{m+n}}{m!\,n!} \\
&= K^2 e^{-\mu x}x^{2\alpha-1}\,S^{0:1;1}_{1:0;0}\left(\begin{array}{c} -\colon [\alpha\colon \kappa]\,;[\alpha\colon \kappa] \\ \\ \big[2\alpha\colon \kappa,\kappa\big]\colon -\,;- \end{array}\middle|\,\begin{array}{c} -ax^\kappa \\ -ax^\kappa \end{array}\right).
\end{aligned} \tag{12}$$

Of course, for $x \leq 0$, the density $\varphi(x)$ terminates. Using (9) and (12) we build easily the PDF, the associated CDF and the related reliability function of the sum $\xi+\eta$ of two i.i.d. r.v.'s having $gW(\theta)$ distribution. Indeed, the distribution function becomes

$$\Phi(x) = K^2 \int_0^x e^{-\mu t} t^{2\alpha-1} S^{0:1;1}_{1:0;0}\left(\begin{array}{c} -\colon [\alpha\colon \kappa]; [\alpha\colon \kappa] \\ \\ [2\alpha\colon \kappa, \kappa]\colon -; - \end{array} \middle| \begin{array}{c} -at^\kappa \\ -at^\kappa \end{array} \right) dt, \tag{13}$$

therefore, taking the above introduced setting, the reliability function of the switched *active* $\leftrightarrow$ *standby* component–couple will be

$$R^{(1)}_{gW}(x) = K^2 \int_x^\infty e^{-\mu t} t^{2\alpha-1} S^{0:1;1}_{1:0;0}\left(\begin{array}{c} -\colon [\alpha\colon \kappa]; [\alpha\colon \kappa] \\ \\ [2\alpha\colon \kappa, \kappa]\colon -; - \end{array} \middle| \begin{array}{c} -at^\kappa \\ -at^\kappa \end{array} \right) dt, \tag{14}$$

remarking that in both last relations $x \geq 0$, while for $x < 0$, $\Phi(x) = 1 - R^{(1)}_{gW}(x) \equiv 0$.

By these facts we prove our next principal result.

Theorem 3. *Let n components, having $gW(\theta), \theta = (\alpha, \mu, a, \kappa) > 0$ life distributions, be connected in series forming a composite system* (S)*, and connected in parallel to form a composite system* (P)*. Improving the pointwise reliability of $1 \leq r \leq n$ components by reduction method and by cold–duplication $1 \leq p \leq n$, the associated pointwise* SEF $\mathbf{r}^C_A(x|gW)$ *and the related pointwise survival equivalence factors $\rho^C_{A,gW}, A \in \{\mathsf{S}, \mathsf{P}\}$ are given by*

$$\begin{aligned} \mathbf{r}^C_{\mathsf{S}}(x|gW) &= \left[\frac{R_{gW}(x\rho^C_{\mathsf{S},gW})}{R_{gW}(x)} \right]^p, \\ \rho^C_{\mathsf{S},gW} &= x^{-1} R^{-1}_{gW}\Big(\big[R_{gW}(x)\big]^{1-p/r} \big[R^{(1)}_{gW}(x)\big]^{p/r} \Big); \\ \mathbf{r}^C_{\mathsf{P}}(x|gW) &= \frac{1 - [1 - R_{gw}(x\rho^C_{\mathsf{P},gW})]^p [1 - R_{gW}(x)]^{n-p}}{1 - [1 - R_{gW}(x)]^n}, \\ \rho^C_{\mathsf{P},gW} &= x^{-1} R^{-1}_{gW}\Big(1 - \big[1 - R_{gW}(x)\big]^{1-p/r} \big[R^{(1)}_{gW}(x)\big]^{p/r} \Big) \qquad (x > 0). \end{aligned}$$

Here $R_{gW}(x)$ is given in (10) *and R^{-1}_{gW} is its inverse R_{gW}; while $R^{(1)}_{gW}(x)$, the reliability function of* working $\leftrightarrow$ standby *cold–duplication components pair, is given by* (14).

4 Estimating ρ^C close to the origin

The building blocks of the reliability functions R_{gW} and $R^{(1)}_{gW}$, that is for the survival functions $\mathscr{S}_{gW,\mathsf{S}}, \mathscr{S}_{gW,\mathsf{P}}$ for the gamma–Weibull distribution are the upper incomplete Gamma function, and *a fortiori* the Srivastava–Daoust S–function. Therefore to establish their asymptotics when $x \to 0^+$, we need the following auxiliary result.

Here, and in what follows, the Landau's $\mathscr{O}$–notation is used, that is $f = \mathscr{O}(g)$ near to some x_0 means that there exists some absolute constant M for which $|f/g| \leq M$ for all x in the neighborhood of x_0.

Lemma 1. *For all $s, B, b > 0$, $z \to 0^+$ we have*

$$ {}_1\psi_0\left[\begin{array}{c|c} (B,b) & \\ \overline{\quad} & \end{array}\right]^{\Gamma}(s,z) = {}_1\Psi_0\left[\begin{array}{c|c} (B,b) & s \\ \overline{\quad} & \end{array}\right] - \frac{z^B}{B} + \mathscr{O}(z^{B+1}). \tag{15} $$

Proof. Having in mind the expansion

$$ \Gamma(s,z) = \Gamma(s) - \frac{z^s}{s} + \mathscr{O}\big(z^{s+1}\big) \qquad (s > 0, z \to 0), $$

which follows by [1, p. 197, Eqs. (4.4.5–6)], we have

$$ \begin{aligned} H = {}_1\psi_0\left[\begin{array}{c|c} (B,b) & \\ \overline{\quad} & \end{array}\right]^{\Gamma}(s,z) &= \sum_{n=0}^{\infty} \Gamma(B+bn,z)\,\frac{s^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[\Gamma(B+bn) - \frac{z^{B+bn}}{B+bn} + \mathscr{O}(z^{B+bn+1})\right]\frac{s^n}{n!} \\ &= {}_1\Psi_0\left[\begin{array}{c|c} (B,b) & s \\ \overline{\quad} & \end{array}\right] - \frac{z^B}{b}\sum_{n=0}^{\infty}\frac{(sz^b)^n}{(B/b+n)\,n!} + \mathscr{O}\left(\sum_{n=0}^{\infty}\frac{z^{B+bn+1}}{n!}\right) \end{aligned} $$

Since

$$ \frac{1}{B/b+n} = \frac{\Gamma(B/b+n)}{\Gamma(B/b+1+n)} = \frac{b}{B}\,\frac{(B/b)_n}{(B/b+1)_n} $$

where the Pochhammer symbol

$$ (\beta)_m = \frac{\Gamma(\beta+m)}{\Gamma(\beta)} = \beta(\beta+1)\cdots(\beta+m-1), \quad \big(m \in \mathbb{N}, \beta \in \mathbb{C}\big), $$

while we take by convention that $(0)_0 = 1$, and in terms of the confluent hypergeometric function

$$ {}_1F_1\left[\begin{array}{c|c} a & t \\ c & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\,\frac{t^n}{n!}, $$

expressing

$$ \sum_{n=0}^{\infty}\frac{(B/b)_n\,(sz^b)^n}{(B/b+1)_n\,n!} = {}_1F_1\left[\begin{array}{c|c} B/b & sz^b \\ B/b+1 & \end{array}\right], $$

we get

$$ H = {}_1\Psi_0\left[\begin{array}{c|c} (B,b) & s \\ \overline{\quad} & \end{array}\right] - \frac{z^B}{B}\,{}_1F_1\left[\begin{array}{c|c} B/b & sz^b \\ B/b+1 & \end{array}\right] + \mathscr{O}\left(z^{B+1}\mathrm{e}^{sz^b}\right). $$

Knowing that ${}_1F_1[\cdot|sz^b] = 1 + \mathscr{O}(z^b)$, moreover $\mathrm{e}^{sz^b} = 1 + \mathscr{O}(z^b)$ when z approaches zero, (15) is proved. □

To determine the asymptotics of $R_{gW}(x)$ for small positive x from (10), it is enough to apply (15) from Lemma to the numerator expressions. So the

Lemma 2. *For all* $\theta = (\alpha, \mu, a, \kappa) > 0$ *and* $x \to 0^+$ *we have*

$$R_{gW}(x) = 1 - \frac{K}{\alpha} x^\alpha + \mathscr{O}\left(x^{\alpha+\max(1,\kappa)}\right). \tag{16}$$

Proof. Assume $\kappa > 1$. Direct application of (15) to the appropriate case in (10) results in

$$\begin{aligned} R_{gW}(x) &= \frac{{}_1\psi_0\left[\begin{array}{c|c} (\alpha/\kappa, 1/\kappa) & \\ \overline{} & \end{array}\right]^{\Gamma}\!\left(-\mu a^{-1/\kappa}, ax^\kappa\right)}{{}_1\Psi_0\left[\begin{array}{c|c} (\alpha/\kappa, 1/\kappa) & -\dfrac{\mu}{a^{1/\kappa}} \\ \overline{} & \end{array}\right]} \\ &= 1 - \frac{\kappa\,(ax^\kappa)^{\alpha/\kappa}}{\alpha\,{}_1\Psi_0\left[\begin{array}{c|c} (\alpha/\kappa, 1/\kappa) & -\dfrac{\mu}{a^{1/\kappa}} \\ \overline{} & \end{array}\right]} + \mathscr{O}\left([x^\kappa]^{\alpha/\kappa+1}\right) \\ &= 1 - \frac{K}{\alpha} a x^\alpha + \mathscr{O}\left([x^\kappa]^{\alpha/\kappa+1}\right), \end{aligned}$$

which is exactly the considered case in (16). We handle the remaining two cases, $\kappa \in (0,1)$ and $\kappa = 1$ in the same way. ☐

Lemma 3. *For all* $\theta = (\alpha, \mu, a, \kappa) > 0$ *and* $x \to 0^+$ *we have*

$$R_{gW}^{(1)}(x) = 1 - \frac{K^2\,\Gamma^2(\alpha)}{\Gamma(2\alpha+1)} x^{2\alpha} + \mathscr{O}(x^{2\alpha+\kappa}). \tag{17}$$

Proof. Consider the first three addends in series:

$$S^{0:1;1}_{1:0;0}\left(\begin{array}{c|c} -\colon [\alpha\colon \kappa]; [\alpha\colon \kappa] & -ax^\kappa \\ [2\alpha\colon \kappa, \kappa]\colon -; - & -ax^\kappa \end{array}\right) = s_0 + s_1 x^\kappa + \mathscr{O}(x^{2\kappa}),$$

where

$$s_0 = \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}, \qquad s_1 = -2a\,\frac{\Gamma(\alpha)\Gamma(\alpha+\kappa)}{\Gamma(2\alpha+\kappa)}.$$

Thus the PDF (12) and the CDF (13) behave like

$$\begin{aligned} \varphi(x) &= \frac{K^2\,\Gamma^2(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} + \mathscr{O}(x^{2\alpha+\kappa-1}) \\ \Phi(x) &= \int_0^x \varphi(t)\,\mathrm{d}t = \frac{K^2\,\Gamma^2(\alpha)}{\Gamma(2\alpha+1)} x^{2\alpha} + \mathscr{O}(x^{2\alpha+\kappa}), \end{aligned}$$

which confirms the stated expansion. ☐

Now, we are ready to obtain the factors ρ^C in function of the variable $x \to 0^+$.

Theorem 4. *For all $\theta = (\alpha, \mu, a, \kappa) > 0$ and $x \to 0^+$ the p–cold duplicated components, r–equivalence reduction improved pointwise survival equivalence factor in the case of series composite system, is equal to*

$$\rho^C_{\mathsf{S},gW} = \left(1 - \frac{p}{r} + \frac{pK\Gamma(\alpha)\Gamma(\alpha+1)}{r\Gamma(2\alpha+1)}\,x^\alpha + \mathscr{O}(x^{2\alpha})\right)^{1/\alpha}. \tag{18}$$

Moreover, p–cold duplicated, r–equivalence reduction improved pointwise survival equivalence factor in the case of parallel composite system will be

$$\rho^C_{\mathsf{P},gW} = x^{-p/r}\left(\frac{\alpha}{K}\right)^{p/(\alpha r)}\left(1 - \frac{pK^2\Gamma^2(\alpha)}{r\Gamma(2\alpha+1)}\,x^{2\alpha} + \mathscr{O}(x^{2\alpha+\kappa})\right)^{1/\alpha}. \tag{19}$$

Proof. We have to solve the equations

$$\mathscr{S}^\rho_{gW,\mathsf{S}_r}(x) = \mathscr{S}^C_{gW,\mathsf{S}_p}(x), \qquad \mathscr{S}^\rho_{gW,\mathsf{P}_r}(x) = \mathscr{S}^C_{gW,\mathsf{P}_p}(x)$$

in ρ for some enough small positive fixed x.

Knowing the constraint $p \le r$ for the series system (S_r), the first equation one reduces *via* (3) to

$$R_{gW}(\rho x) = \big[R_{gW}(x)\big]^{1-p/r}\,[R^{(1)}_{gW}(x)]^{p/r}.$$

In turn, applying Lemmata 2 and 3 we obtain

$$\begin{aligned} 1 - \frac{K}{\alpha}(\rho x)^\alpha + \mathscr{O}(x^{\alpha+M}) &= \left[1 - \frac{K}{\alpha}x^\alpha + \mathscr{O}(x^{\alpha+M})\right]^{1-p/r} \\ &\quad \times \big[1 - C^{(1)}_{gW}x^{2\alpha} + \mathscr{O}(x^{2\alpha+\kappa})\big]^{p/r}, \end{aligned}$$

where

$$M = \max\{1, \kappa\}, \qquad C^{(1)}_{gW} = \frac{K^2\Gamma^2(\alpha)}{\Gamma(2\alpha+1)}.$$

Therefore

$$\rho^C_{\mathsf{S},gW} = \left(1 - \frac{p}{r} + \frac{pK\Gamma(\alpha)\Gamma(\alpha+1)}{r\Gamma(2\alpha+1)}\,x^\alpha + \mathscr{O}(x^{2\alpha})\right)^{1/\alpha}.$$

The parallel connected system (P_r) with r reduction–improved and p cold duplicated components will have the same survival function value at some fixed time x, when the second equation, reads $\mathscr{S}^\rho_{gW,\mathsf{P}_r}(x) = \mathscr{S}^C_{gW,\mathsf{P}_p}(x)$ holds true; it can be rewritten into

$$R_{gW}(\rho x) = 1 - \big[1 - R_{gW}(x)\big]^{1-p/r}\,[R^{(1)}_{gW}(x)]^{p/r}.$$

However, for vanishing x, in turn, this equation one can transform into

$$1-\frac{K}{\alpha}(\rho x)^{\alpha}+\mathscr{O}(x^{\alpha+M})=1-\left[\frac{K}{\alpha}x^{\alpha}+\mathscr{O}(x^{\alpha+M})\right]^{1-p/r} \times\left[1-C_{gW}^{(1)}x^{2\alpha}+\mathscr{O}(x^{2\alpha+\kappa})\right]^{p/r},$$

which solution is

$$\rho_{\mathsf{P},gW}^{C}=x^{-p/r}\left(\frac{\alpha}{K}\right)^{p/(\alpha r)}\left(1-\frac{pK^2\Gamma^2(\alpha)}{r\Gamma(2\alpha+1)}x^{2\alpha}+\mathscr{O}(x^{2\alpha+\kappa})\right)^{1/\alpha}.$$

The proof is completed. □

Remark 2. From (18) follows that

$$\lim_{x\to 0^+}\rho_{\mathsf{S}}^{C}=\left(1-\frac{p}{r}\right)^{1/\alpha}=L.$$

However, this result shows that the the gamma–Weibull lifetime distribution cannot guarantee stable series connected system at the functioning beginning when α is small, because $p\le r$ and

$$L=\left(1-\frac{p}{r}\right)^{1/\alpha}\xrightarrow[\alpha\to 0^+]{}0.$$

In some cases, when the parameter α depends on the cold duplicated components, and the size of components have been improved by reduction method, that is $\alpha=\alpha(p,r)$, the quantity L can take positive limit with vanishing α. Indeed, e.g. for $p\asymp\alpha r$, we have $L\to \mathrm{e}^{-1}$, when $\alpha\to 0^+$.

The situation with the growing α is the opposite: L approaches 100%. □

5 Simulation results and conclusion

To illustrate how the theory, which was obtained in the previous sections, can be applied, three different parameter cases are presented in this section.

The $gW(\theta)$ lifetime–distribution's PDF takes three analytically different forms depending on the κ, compare Fig 1. So do the associated reliability functions as illustrates Fig 2. *via* (10). Therefore we decide to study the PDF (6) when $(\alpha,\mu,a)=(3,3,2)$ and $\kappa\in\{0.363,1,2\}$ as shown in Fig 1.

Assume that $(\mathsf{S}),(\mathsf{P})$ consist from $n=8$ IID components, while improving $r=3, p=2$ components by reduction method we get $(\mathsf{S}_3),(\mathsf{P}_2)$ respectively. These systems are now treated by cold duplication. According to Theorem 1 $p\le r=3$ components have to be improved by cold duplication in (S); no such limitation occurs for (P).

The values of normalizing constant K in the considered cases become

$$K(3,3,2,0.363)=84.8514,\ K(3,3,2,1)=62.5000,\ K(3,3,2,2)=45.3513;$$

the calculations where performed by the WolframAlpha|PRO computational engine. However, the parameter $L = 1/\sqrt[3]{3}$ throughout.

The numerical simulation results include three cases: $\kappa \in (0,1), \kappa = 1, \kappa > 1$, presented on Table I, II and III respectively. The tables contain five sampled values of the reliability function $R_{gW}(x)$ of a component, the realibility function $R_{gW}^{(1)}(x)$ of the cold duplicated, switched active $\leftrightarrow$ standby component–couple, the survival functions $\mathscr{S}_{gW,\mathsf{S}}(x)$, $\mathscr{S}_{gW,\mathsf{P}}(x)$ of series and parallel systems respectively; the survival equivalence factors $\rho_\mathsf{S}^H, \rho_\mathsf{P}^H$ all under the same number of components $r = 3, p = 2$ improved by reduction method and by cold duplication respectively, all nearby to the origin. The sample nodes $x = 0.10 + j \cdot 0.05,\ j = \overline{0,4}$ are used in all cases (by comparison purposes). Thus, the not to large argument values $x \ll 1$ enable to approximate all these functional characteristics by Theorem 2, Lemmata 2, 3 and Theorem 4 respectively. More precisely, writing $\widetilde{G}$ for the approximant in the asymptotic expansion $G(x) = \widetilde{G}(x) + \mathscr{O}(x^\upsilon)$, for certain suitable G, υ, the formulae applied read as follows:

$$\widetilde{R}_{gw}(x) = 1 - \frac{K}{\alpha}x^\alpha, \qquad \widetilde{R}_{gw}^{(1)}(x) = 1 - \frac{K^2\,\Gamma^2(\alpha)}{\Gamma(2\alpha+1)}x^{2\alpha}$$

$$\widetilde{\mathscr{S}_{gW,\mathsf{S}}}(x) = \left(1 - \frac{K}{\alpha}x^\alpha\right)^8, \qquad \widetilde{\mathscr{S}_{gW,\mathsf{P}}}(x) = 1 - \left(\frac{K}{\alpha}x^\alpha\right)^8$$

$$\widetilde{\rho}_{\mathsf{S},gW}^C = \left(\frac{1}{3} + \frac{2K\Gamma(\alpha)\Gamma(\alpha+1)}{3\Gamma(2\alpha+1)}x^\alpha\right)^{1/\alpha}$$

$$\widetilde{\rho}_{\mathsf{P},gW}^C = x^{-2/3}\left(\frac{\alpha}{K}\right)^{2/(3\alpha)}\left(1 - \frac{2K^2\,\Gamma^2(\alpha)}{3\,\Gamma(2\alpha+1)}x^{2\alpha}\right)^{1/\alpha}.$$

According to Remark 2, $L \simeq 0.69336$ when $x \to 0^+$ which is visible in all three tables – compare the sixth columns **first data**.

After these SEF simulations the considered models' $(\mathsf{S}), (\mathsf{P})$ survival functions expose the full meanings of Theorems 2, 3 and 4, where the IID components reliability function is, for the first time, applied to the gamma–Weibull distribution, since $gW(\theta)$ generalizes Gamma–distribution [18] and the various topology composite systems for the exponential $\mathscr{E}(\lambda)$ lifetime–distribution studied by Sarhan [11, 12, 13, 14]. The simulations were realized near to the origin, which show that the asymptotics is polynomial in all cases. Accordingly, the cold duplication can be successfully replaced by reliability reduction method using the *at most the same number of improved components* for (S), while the reduction method is *independent of cold duplication* in the case of parallel systems (P). Also, it would be of considerable interest to connect our results and/or extend it to another fashion questions discussed e.g. in the recent paper by Morariu and Zaharia, see [5].

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Table 1
Numerical simulation results for $\theta = (3,3,2,0.363)$.

x	$\widetilde{R}_{gW}$	$\widetilde{R}^{(1)}_{gW}$	$\widetilde{\mathscr{S}}_{gW,\mathsf{S}}$	$\widetilde{\mathscr{S}}_{gW,\mathsf{P}}$	$\widetilde{\rho}^{C}_{\mathsf{S},gW}$	$\widetilde{\rho}^{C}_{\mathsf{P},gW}$
0.10	0.97172	0.99996	0.79491	1.00000	**0.69401**	2.20851
0.15	0.90454	0.99954	0.44816	1.00000	0.69556	1.68525
0.20	0.77373	0.99744	0.12844	1.00000	0.69855	1.39049
0.25	0.55807	0.99024	0.00941	0.99855	0.70343	1.19637
0.30	0.23634	0.97084	9.733E-6	0.88433	0.71058	1.05482

Table 2
Numerical simulation results for $\theta = (3,3,2,1)$.

x	$\widetilde{R}_{gW}$	$\widetilde{R}^{(1)}_{gW}$	$\widetilde{\mathscr{S}}_{gW,\mathsf{S}}$	$\widetilde{\mathscr{S}}_{gW,\mathsf{P}}$	$\widetilde{\rho}^{C}_{\mathsf{S},gW}$	$\widetilde{\rho}^{C}_{\mathsf{P},gW}$
0.10	0.97917	0.99998	0.84499	1.00000	**0.69384**	2.46478
0.15	0.92969	0.99975	0.55808	1.00000	0.69498	1.80381
0.20	0.83333	0.99861	0.23257	1.00000	0.69719	1.48864
0.25	0.67488	0.99470	0.04283	0.99987	0.70081	1.28175
0.30	0.43750	0.98418	0.00134	0.98998	0.70613	1.13238

Table 3
Numerical simulation results for $\theta = (3,3,2,2)$.

x	$\widetilde{R}_{gW}$	$\widetilde{R}^{(1)}_{gW}$	$\widetilde{\mathscr{S}}_{gW,\mathsf{S}}$	$\widetilde{\mathscr{S}}_{gW,\mathsf{P}}$	$\widetilde{\rho}^{C}_{\mathsf{S},gW}$	$\widetilde{\rho}^{C}_{\mathsf{P},gW}$
0.10	0.98489	0.99999	0.88527	1.00000	**0.69371**	2.53841
0.15	0.94898	0.99990	0.65774	1.00000	0.69454	1.93712
0.20	0.87906	0.99927	0.35658	1.00000	0.69615	1.59884
0.25	0.76380	0.99721	0.11583	1.00000	0.69878	1.37721
0.30	0.59184	0.99167	0.01505	0.99923	0.70267	1.21808

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