An Application of the Interpretation Method in the Axiomatization of the Lukasiewicz Logic and the Product Logic

Aleksandar Perović, Maja Jovanović, Aleksandar Jovanović

Group for Intelligent Systems, Faculty of Mathematics
Vojvode Stepe 305, 11000 Belgrade, Serbia
pera@sf.bg.ac.yu

Abstract. During the last two decades, Group for intelligent systems at Mathematical faculty in Belgrade has developed several theorem provers for different kind of formal systems. Lately, we have turned our attention to fuzzy logic and development of the corresponding theorem prover. The first step is to find the suitable axiomatization, i.e., the formalization of fuzzy logic that is sound, complete and decidable. It is well known that there are fuzzy logics (such as Product logic) that require infinitary axiomatization in order to tame the non-compactness phenomena. Though such logics are strongly complete (every consistent set of formulas is satisfiable), the only possible decidability result is the satisfiability of a formula. Therefore, we have adapted the method of Fagin, Halpern and Megiddo for polynomial weight formulas in order to interpret the Lukasiewicz and the Product logic into the first order theory of the reals.

1 Introduction

The fuzzy logic emerged in mid sixties of the 20th century in order to mathematically capture the notion of uncertainty and enable mathematical tools for reasoning about notions with inherited fuzziness, such as being tall, young, fat, bald etc. The new semantics involve t-norms as ‘and’ operators, and s-norms as ‘or’ operators. A t-norm is any function $T:[0,1]^2 \to [0,1]$ such that:

- $T(x,y) = T(y,x)$
- $T(x,T(y,z)) = T(T(x,y),z)$
- $T(x,1) = x$
- $T(x,z) \le T(y,z)$ whenever $y \le z$.

The corresponding s-norm S is defined by $S(x,y)=1-T(1-x,1-y)$. For instance, the truth evaluation of the Product conjunction and negation is defined by the following two clauses:

- $e(\alpha \wedge \beta)=e(\alpha)e(\beta)$
- $e(\neg\alpha)=1$ if $e(\alpha)=0$, otherwise $e(\alpha)=0$.

The underlying t-norm is the product norm $T(x,y)=xy$.

Development of formal systems for fuzzy logic is a well worked area (see [1] and [3]). The main goal of any axiomatization is to achieve some variant of completeness. For our purposes, two of those are of interest:

- *Simple completeness*: a formula α is a theorem iff it is valid (satisfied in each model).
- *Strong completeness*: every consistent set of formulas has a model.

For more on the completeness and other basic model theoretical notions we refer the reader to [3] and [4]. Some fuzzy logics, such as $L\Pi\frac{1}{2}$ logic, one axiomatization of the Lukasiewicz logic and the Product logic, are only simply complete. Detailed treatment and the axioms of $L\Pi\frac{1}{2}$ can be found in [1]. In the case of $L\Pi\frac{1}{2}$ we can define a consistent theory that resembles the type of a proper infinitesimal ($\varepsilon > 0$ is a proper infinitesimal if $n\varepsilon < 1$ for all $n = 1,2,3,\ldots$):

$$\Sigma = \{\neg_{\Pi}(p \rightarrow_{\Pi} 0)\} \cup \left\{ p \rightarrow_{\Pi} \frac{1}{n} : n = 1,2,3,\ldots \right\}.$$

Namely, theory Σ says that the truth value of p is greater than 0 and lesser than each $\frac{1}{n}$, so it must be a proper infinitesimal. Hence, Σ is unsatisfiable, i.e., there is no Archimedean truth evaluation $e : For \rightarrow [0,1]$ such that $e(\alpha)=1$ for all $\alpha \in \Sigma$. However Σ is a consistent set of formulas in $L\Pi\frac{1}{2}$. Consequently,

$L\Pi\frac{1}{2}$ is not strongly complete logic. A strongly complete axiomatization of the Lukasiewicz logic and the Product logic can be found in [1] and [3].

The main result of this paper is the application of the interpretation method in the axiomatization of the Lukasiewicz logic and the Product logic. Namely, we have interpreted those two fuzzy logics in the first-order theory of real closed fields (RCF). Our methodology is similar to the one described in [2]. Introduced interpretation allows development of a theorem prover for the Lukasiewic logic and the Product logic that is within PSPACE (polynomial complexity).

The rest of the paper is organized as follows: real-valued propositional logics are discussed in Section 2; Section 3 introduces the interpretation of the Lukasiewicz logic and the Product logic in the theory of real-closed fields; concluding remarks are in Section 4.

2 Real-valued Propositional Logics

First we will build a formal propositional language, then semantically define the notion of the real-valued propositional logic, and, finally, say something about axiomatization of such logics. Like in any formal language, we will start with some basic symbols, and then define the word-formation rules which will be applied in the recursive construction of formulas.

Our basic symbols are propositional letters, truth constants and unary and binary connectives. The set of all propositional letters will be denoted by P, while its elements will be denoted by p, q and r, indexed if necessary. The truth constants will be denoted by c_s, where s is any rational number from the real unit interval $[0,1]$. The unary connectives will be denoted by U, indexed or primed, while the binary connectives will be denoted by B, indexed if necessary. The set For of propositional formulas is recursively defined as follows:

- Propositional letters and truth constants are propositional formulas.
- If α is a propositional formula and U is a unary connective, then $U\alpha$ is a propositional formula.
- If α, β are propositional formulae and B is a binary connective, then $(\alpha B \beta)$ is a propositional formula.
- Propositional formulae can be obtained only by the finite application of the above steps.

A real-valued propositional logic (RVPL) is any function $\Lambda : [0,1]^P \to [0,1]^{For}$ with the following properties:

1 $\Lambda f(p) = f(p)$ for all $f \in [0,1]^P$ and all $p \in P$.

2 $\Lambda f(c_s) = s$ for all $f \in [0,1]^P$ and all c_s.

A propositional formula α is Λ-valid if $\Lambda f(\alpha) = 1$ for all $f \in [0,1]^P$. A RVPL Λ is a truth-functional with respect to the unary connective U if there is a function $F_U : [0,1] \to [0,1]$ such that

$$\Lambda f(U\alpha) = F_U(\Lambda f(\alpha))$$

for all $\alpha \in For$ and all $f \in [0,1]^P$. Similarly, Λ is truth-functional with respect to the binary connective B if there is a function $F_B : [0,1]^2 \to [0,1]$ such that

$$\Lambda f(\alpha B \beta) = F_B(\Lambda f(\alpha), \Lambda f(\beta))$$

for all $\alpha, \beta \in For$ and all $f \in [0,1]^P$. Every fuzzy logic is a RVPL that is truth-functional with respect to some finite set of connectives.

The Lukasiewicz-Product logic is a RVPL $\Lambda_{L\Pi}$ that is truth functional with respect to the binary connectives $\wedge_\Pi$ (Product conjunction), $\to_\Pi$ (Product implication) and $\to_L$ (Lukasiewicz implication), where:

- $F_{\wedge_\Pi}(x, y) = xy$.
- $F_{\to_\Pi}(x, y) = 1$ if $x \le y$, otherwise $F_{\to_\Pi}(x, y) = \dfrac{y}{x}$.
- $F_{\to_L}(x, y) = \min(1 - x + y, 1)$.

For the sake of simplicity, we may assume that the above three connectives are the only connectives.

Next we will turn to the syntactical propositional logics (SPL's). A SPL Γ is a pair $\langle A_\Gamma, R_\Gamma \rangle$, where A_Γ (the set of axioms) is a subset of For, while R_Γ (the set of derivation rules) is a subset of the set of all partial functions from the power set of For to For. A proof in Γ is any sequence S of propositional formulae with the following properties:

- The order type of S is a successor ordinal.

- For each S_ξ (S_ξ is the ξ-th member of S), $S_\xi \in A_\Gamma$ or $S_\xi = F(X)$, where $F \in R_\Gamma$ and $X \subseteq \{S_\eta : \eta < \xi\}$.

A formula α is a theorem of Γ if it is the last member of some proof in Γ. A SPL Γ is an axiomatization of a RVPL Λ if, for all $\alpha \in For$, α is a theorem of Γ iff α is Λ-valid. For instance, $L\Pi\frac{1}{2}$ is an axiomatization of $\Lambda_{L\Pi}$.

3 Interpretation of $L\Pi\frac{1}{2}$ in RCF

We will assume that the only connectives appearing in propositional formulae are Product conjunction, Product implication and Lukasiewicz implication. Let $L_{OF} = \{+,\cdot,\leq,0,1\}$ (i.e. L_{OF} is a first order language of the theory of ordered fields). As it is usual, by RCF we will denote the L_{OF}-theory of the real closed fields. The axioms of RCF can be found in [4]. Here we will just say that every model of RCF (in the sense of the first order predicate logic, see [4, 6]) is an ordered field in which every polynomial of the odd degree has a root.

By $For_{L\Pi}$ we will denote the set of all propositional formulae built over the countable set of propositional letters and the set of the truth constants $\{c_s : s \in [0,1] \cap Q\}$ by means of $\wedge_\Pi$ (Product conjunction), $\to_\Pi$ (Product implication) and $\to_L$ (Lukasiewicz implication). In other words, $For_{L\Pi}$ is the set of all formulas of the Lukasiewicz – Product fuzzy logic. Our aim is to interpret this logic in RCF (for the necessary background on the interpretation method we refer the reader to [6]).

First of all, we will extend the L_{OF} with the countably many new constant symbols C_α, where $\alpha \in For_{L\Pi}$. The intended meaning of C_α is to represent the truth value of α. To provide this, we will extend the theory RCF with the following axioms (in the first order predicate calculus):

- $C_{\alpha \to_L \beta} = \min(1 - C_\alpha + C_\beta, 1)$.
- $C_{\alpha \wedge_\Pi \beta} = C_\alpha \cdot C_\beta$.
- $C_\alpha \leq C_\beta \to C_{\alpha \to_\Pi \beta} = 1$.

- $C_\beta < C_\alpha \to C_\alpha = C_\beta \cdot C_{\alpha \to_\Pi \beta}$.
- $0 \le C_p \le 1$.
- $C_{c_s} = s$.

The above axioms actually follow the standard definition of the truth evaluation in the case of the Lukasiewicz implication, Product conjunction and Product implication. The last axiom provides the usual behavior of the truth constants (i.e. they are, up to equivalence, rational numbers between 0 and 1. Obtained first order theory will be denoted by $RCF_{L\Pi}$.

Using the compactness theorem for the first order predicate logic, one can easily show that $RCF_{L\Pi}$ is a consistent first order theory. It is well known that α is a theorem of $L\Pi\frac{1}{2}$ (see [1, 3]) if and only if α is $\Lambda_{L\Pi}$-valid. An immediate consequence of the definition of $RCF_{L\Pi}$ is the fact that α is $\Lambda_{L\Pi}$-valid if and only if $C_\alpha = 1$ is a theorem of $RCF_{L\Pi}$. Thus, we have interpreted the logic $L\Pi\frac{1}{2}$ in the theory $RCF_{L\Pi}$. It remains to interpret $RCF_{L\Pi}$ in RCF.

Here we will give only the sketch of the proof. The detailed proof of this fact would be given elsewhere. The axioms for new constants provide the following fact: for each $\alpha \in For_{L\Pi}$, there is an RCF-definable function symbol F such that the formula

$$C_\alpha = F\left(C_{p_1}, \ldots, C_{p_n}\right),$$

where $p_1, \ldots, p_n$ are all propositional letters appearing in α. Thus, each sentence of the extended language is $RCF_{L\Pi}$ equivalent to some sentence of the form $\phi\left(C_{p_1}, \ldots, C_{p_n}\right)$. Finally, such a sentence $\phi\left(C_{p_1}, \ldots, C_{p_n}\right)$ is a theorem of $RCF_{L\Pi}$ if and only if the sentence

$$\exists x_1 \ldots \exists x_n \left(0 \le x_1 \le 1 \wedge \cdots \wedge 0 \le x_n \le 1 \wedge \phi\left(x_1, \ldots, x_n\right)\right)$$

is a theorem of RCF. Thus, we have interpreted the Lukasiewicz – Product logic into the first order theory of the reals.

Conclusion

It is a well known fact that the first order theory of the reals is decidable. Though the general decision procedure for RCF is in EXPSPACE, the satisfiability of $\alpha \in For_{L\Pi}$ can be decided by a PSPACE procedure. Namely, the satisfiability of $\alpha \in For_{L\Pi}$ can be equivalently reduced to the existence of the solution of a system of polynomial inequalities. The later problem can be expressed as a purely existential sentence - the universal quantifier does not appear in it in any form (implicit or explicit). As it is shown in [0], this can be decided by the procedure that is in PSPACE.

Our future work will include the implementation of a PSPACE procedure for the existential theory of the reals as well as the construction of the theorem prover for the Lukasiewicz – Product logic based on it.

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