Computation of Boundary Layers

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Abstract:This paper is the first part of a series of studies where we examine several methods for the solution of the boundary layer equation of the fluid mechanics. The first of these is the analytical or rather quasi analytical method due to Blasius. This method reduces a system of partial differential equations to a system of ordinary differential equations and these in turn are solved by numerical methods since no exact solution of the Blasius type equations is known. We determind all the Blasius equation neccessary for up to 11-th order approximation. Our further aim to study the finite difference numerical solutions of the boundary layer equation and some of the methods applying weighted residual principles and by comparing these with the "exact" solutions arrived at by Blasius method develop a quick reliable method for solving the boundary layer equation.

Keywords: Boundary Layer, Blasius Method, Boundary Layer equation

1 Boundary Layer

The motion of a fluid around a solid body according to Prandtl (1904) can be described by the Euler equation of the perfect (that is nonviscous) fluid motion except in a thin layer near the surface of the solid body where the speed of the motion increases from zero to the speed that would be in case if the fluid had no viscosity at all. Outside the boundary layer the fluid may be considered as nonviscous. This is the case when the velocity of the fluid in the direction of the flow around the body increases. When it decreases that is the pressure increases, often the fluid motion unable to follow the bodies' surface and it gets detached and the space between the surface of the solid and the detached fluid is filled with irregularly moving fluid. Prandt's theory of boundary layer, more precisely his equations describing the motion within the boundary layer can predict the point(s) of detachment accurately. The detachment begins where the curve of the velocity profile starts out perpendiclar to the surface of the solid. After this point a backward flow develops. The typical values used for describing the boundary layer are:

δ : Boundary layer thickness is the distance measured from the surface of the solid where the speed of the fluid is within 1% of the speed outside of the boundary layer.

δ_1 : Displacement thickness $\delta_1 = \frac{1}{U}\int_0^\infty (U-u)dy$

δ_2 : Impulseloss thickness $\delta_2 = \frac{1}{U^2}\int_0^\infty (U-u)udy$

δ^{**} : Energyloss thickness $\delta^{**} = \frac{1}{U^3}\int_0^\infty (U-u)^2 udy$

Profile parameter $H_{1,2} = \frac{\delta^{**}}{\delta_2}$

2 The Eqations of the Boundary Layer Flow

Inside the boundary layer that is in the vicinity of the body the forces due to viscosity are comparable in magnitude with the forces of inertia they can however be neglected outside of it. The pressure in the boundary layer could be taken as constant and its value equal to the pressure belonging to the corresponding perfect fluid flow, that is the pressure outside of the boundary layer. Without going into more details we give the equations of the boundary layer motion in case of two dimensional staionary incompressible fluid flow:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where u is the velocity of the fluid in the boundary layer parallel to the tangent of the surface of the solid v is perpendicular to it and U is the velocity outside of the boundary layer. u and v must also satisfy the boundary conditions: $y = 0$: $u = 0$ $v = 0$ and at $y = \infty$ $u = U(x)$.

3 Blasius' Method for Solving the Boundary Layer Equation

The boundary layer equations can be reduced to an infinite system of ordinary differential equations with the following method due to Blasius. By substituting for the velocity components $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, where ψ is the stream function of Lagrange the second of the two boundary layer equations is automaticly satisfied and the first one becomes:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = U \frac{dU}{dx} + \nu \psi_{yyy}.$$

This equation then can be reduced to a set of ordinary differential equation if for Ψ in case of symmetric bodies the following power series expansion is substituted:

$$\psi = \sqrt{\frac{\nu}{u_1}} \{u_1 x f_1(\eta) + 4u_3 x^3 f_3(\eta) + 6u_5 x^5 f_5(\eta) + 8u_7 x^7 f_7(\eta) +$$
$$+ 10u_9 x^9 f_9(\eta) + 12u_{11} x^{11} f_{11}(\eta) + ...$$

where $\eta = y\sqrt{\frac{u_1}{\nu}}$ .and $u_1, u_3, \cdots$ are the coefficients in the power series expansion of $U(x)$, that is:

$$U = u_1 x + u_3 x^3 + u_5 x^5 + u_7 x^7 + u_9 x^9 + u_{11} x^{11} + u_{13} x^{13} + u_{15} x^{15} +$$
$$+ u_{17} x^{17} + u_{19} x^{19} + ...$$

From the last two equations it follows that:

$$U \frac{dU}{dx} = u_1^2 x + 4u_1 u_3 x^3 + \left(6u_1 u_5 + 5u_3^2\right)x^5 + \left(8u_1 u_7 + 8u_3 u_5\right)x^7 +$$
$$\left(10u_1 u_9 + 10u_3 u_7 + 5u_5^2\right)x^9 + \left(12u_1 u_{11} + 12u_3 u_9 + 12u_5 u_7\right)x^{11} + ...$$

$$\psi_x = \sqrt{\frac{\nu}{u_1}}\{u_1 f_1(\eta) + 12u_3x^2 f_3(\eta) + 30u_5x^4 f_5(\eta) + 56u_7x^6 f_7(\eta) + 90u_9x^8 f_9(\eta) +$$
$$+132u_{11}x^{10} f_{11}(\eta) + ...\}.$$

$$\psi_y = u_1 x f_1'(\eta) + 4u_3x^3 f_3'(\eta) + 6u_5x^5 f_5'(\eta) + 8u_7x^7 f_7'(\eta) + 10u_9x^9 f_9'(\eta) +$$
$$+12u_{11}x^{11} f_{11}'(\eta) + ...$$

$$\psi_{xy} = u_1 f_1'(\eta) + 12u_3x^2 f_3'(\eta) + 30u_5x^4 f_5'(\eta) + 56u_7x^6 f_7'(\eta) + 90u_9x^8 f_9'(\eta) +$$
$$+132u_{11}x^{10} f_{11}'(\eta) + ...$$

$$\psi_{yy} = \sqrt{\frac{u_1}{\nu}}\{u_1 x f_1''(\eta) + 4u_3x^3 f_3''(\eta) + 6u_5x^5 f_5''(\eta) + 8u_7x^7 f_7''(\eta) +$$
$$+10u_9x^9 f_9''(\eta) + 12u_{11}x^{11} f_{11}''(\eta) + ...$$

$$\psi_{yyy} = \frac{u_1}{\nu}(u_1 x f_1'''(\eta) + 4u_3x^3 f_3'''(\eta) + 6u_5x^5 f_5'''(\eta) + 8u_7x^7 f_7'''(\eta) +$$
$$+10u_9x^9 f_9'''(\eta) + 12u_{11}x^{11} f_{11}'''(\eta) + ...)$$

and

$$\psi_y\psi_{xy} = u_1^2 x f_1'^2 + 12u_1u_3 f_1' f_3' x^3 + (36u_1u_5 f_1' f_5' + 48u_3^2 f_3'^2)x^5 +$$
$$(64u_1u_7 f_1' f_7' + 192u_3u_5 f_3' f_5')x^7 +$$
$$(100u_1u_9 f_1' f_9' + 320u_3u_7 f_3' f_7' + 180u_5^2 f_5'^2)x^9 +$$
$$(144u_1u_{11} f_1' f_{11}' + 480u_3u_9 f_3' f_9' + 576u_5u_7 f_5' f_7')x^{11} + ...$$

$$\psi_x\psi_{yy} = u_1^2 f_1 f_1'' x + (4u_1u_3 f_1 f_3'' + 12u_1u_3 f_3 f_3'')x^3 +$$
$$(6u_1u_5 f_1 f_5'' + 48u_3^2 f_3 f_3'' + 30u_1u_5 f_5 f_1'')x^5 +$$
$$(8u_1u_7 f_1 f_7'' + 72u_3u_5 f_3 f_5'' + 120u_3u_5 f_5 f_3'' + 56u_1u_7 f_7 f_1'')x^7 +$$
$$(10u_1u_9 f_1 f_9'' + 96u_3u_7 f_3 f_7'' + 180u_5^2 f_5 f_5'' + 224u_3u_7 f_7 f_3'' + 90u_1u_9 f_9 f_1'')x^9 +$$
$$(12u_1u_{11} f_1 f_{11}'' + 120u_3u_9 f_3 f_9'' + 240u_5u_7 f_5 f_7'' + 336u_5u_7 f_7 f_5'' +$$
$$360u_9u_3 f_9 f_3'' + 132u_{11}u_1 f_9 f_1'')x^{11} + ...$$

finally substituting these into the bondary layer equation:

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = UU_x + \psi_{yyy}$$

and comparing the coefficients of the powers of x we get the differential equations of Blasius:

Comparing the coefficients of x yields:

$$u_1^2 f_1^{'2} - u_1^2 f_1 f_1^{''} = u_1^2 + u_1^2 f_{11}^{'''}$$

and from here we get:

$$f_1^{'2} - f_1 f_1^{''} = 1 + f_1^{'''}$$

From the coefficients of x^3 we get:

$$12u_1u_3 f_1^{'} f_3^{'} - (4u_1u_3 f_1 f_3^{''} + 12u_1u_3 f_1^{''} f_3) = 4u_1u_3 + 4u_1u_3 f_3^{'''}$$

that is:

$$3f_1^{'} f_3^{'} - f_1 f_3^{''} - 3f_1^{''} f_3 = 1 + f_3^{'''}$$

As for x^5:

$$36u_1u_5 f_1^{'} f_5^{'} + 48u_3^2 f_3^{'2} - (6u_1u_5 f_1 f_5^{''} + 48u_3^2 f_3 f_3^{''} + 30u_1u_5 f_1^{''} f_5) =$$
$$6u_1u_5 + 3u_3^2 + 6u_1u_5 f_5^{'''}$$

Dividing by $6u_1u_5$:

$$6f_1^{'} f_5^{'} + 8\frac{u_3^2}{u_1u_5} f_3^{'2} - (f_1 f_5^{''} + 8\frac{u_3^2}{u_1u_5} f_3 f_3^{''} + 5f_1^{''} f_5) = 1 + \frac{1}{2}\frac{u_3^2}{u_1u_5} + f_5^{'''}$$

If we seek f_5 in the form

$f_5 = g_5 + \frac{u_3^2}{u_1u_5} h_5$ we get for g_5 and h_5 the following differentialequations:

$$6f_1^{'} g_5^{'} - f_1 g_5^{''} - 5f_1^{''} g_5 = 1 + g_5^{'''}$$

$$6f_1^{'} h_5^{'} + 8f_3^{'2} - f_1 h_5^{''} - 5f_1^{''} h_5 - 8f_3 f_3^{''} = \frac{1}{2} + h_5^{'''}$$

Case x^7. This case is sufficiently complex to demonstrate the method of finding the Blasius type diffrential equations for the general case that is for x^n where n is an arbitrary odd integer. By comparing the coefficients of x^7 we get:

$$(64u_1u_7f_1^{'}f_7^{'}+192u_3u_5f_3^{'}f_5^{'})-$$
$$(8u_1u_7f_1f_7^{''}+72u_3u_5f_3f_5^{''}+120u_5u_3f_5f_3^{''}+56u_7u_1f_7f_1^{''})=$$
$$(8u_1u_7+8u_3u_5)+8u_1u_7f_7^{'''}$$

Dividing by $8u_1u_7$ gives:

$$(8f_1^{'}f_7^{'}+24\frac{u_3u_5}{u_1u_7}f_3^{'}f_5^{'})-(f_1f_7^{''}+9\frac{u_3u_5}{u_1u_7}f_3f_5^{''}+15\frac{u_3u_5}{u_1u_7}f_5f_3^{''}+7f_7f_1^{''})=$$
$$(1+\frac{u_3u_5}{u_1u_7})+f_7^{'''}$$

Let us seek f_7 in the form $f_7=g_7+\frac{u_3u_5}{u_1u_7}\tilde{h}_7$. Substituting this into the last equation yields for g_7 the following ordinary differential equation:

$$8f_1^{'}g_7^{'}-f_1g_7^{''}-7f_7f_1^{''}=1+g_7^{'''}$$

and for $\tilde{h}_7$

$$8\frac{u_3u_5}{u_1u_7}f_1^{'}\tilde{h}_7^{'}+24\frac{u_3u_5}{u_1u_7}f_3^{'}(g_5^{'}+\frac{u_3^2}{u_1u_5}h_5^{'})-\frac{u_3u_5}{u_1u_7}f_1\tilde{h}_7^{''}-$$
$$9\frac{u_3u_5}{u_1u_7}f_3(g_5^{''}+\frac{u_3^2}{u_1u_5}h_5^{''})-15\frac{u_3u_5}{u_1u_7}f_3^{''}(g_5+\frac{u_3^2}{u_1u_5}h_5)-7\tilde{h}_7f_1^{''}=$$
$$\frac{u_3u_5}{u_1u_7}+\frac{u_3u_5}{u_1u_7}\tilde{h}_7^{'''}$$

Substituting for $\tilde{h}_7$ $\tilde{h}_7=h_7+\frac{u_3^2}{u_1u_7}k_7$ yields for h_7

$$8f_1^{'}h_7^{'}+24f_3^{'}g_5^{'}-f_1h_7^{''}-9f_3g_5^{''}-15f_3^{''}g_5-7f_1^{''}h_7=1+h_7^{'''}$$

and for k_7

$$8f_1^{'}k_7^{'}+24f_3^{'}h_5^{'}-f_1k_7^{''}-9f_3h_5^{''}-15f_3^{''}h_5-7f_1^{''}k_7=k_7^{'''}$$

With the same method we can arrive at the equations for all the f_n Blasius functions. With n increasing f_n has to be broken down into more and more terms. Here the forms of f_n and the corresponding differential equations are given up to the order of 11.

$$f_5 = g_5 + \frac{u_3^2}{u_1 u_5} h_5$$

$$f_7 = g_7 + \frac{u_3 u_5}{u_1 u_7} h_7 + \frac{u_3^2}{u_1^2 u_7} k_7$$

$$f_9 = g_9 + \frac{u_3 u_7}{u_1 u_9} h_9 + \frac{u_5^2}{u_1 u_9} k_9 + \frac{u_3^2 u_5}{u_1^2 u_9} j_9 + \frac{u_3^4}{u_1^3 u_9} q_9$$

$$f_{11} = g_{11} + \frac{u_3 u_9}{u_1 u_{11}} h_{11} + \frac{u_5 u_7}{u_1 u_{11}} k_{11} + \frac{u_3^2 u_7}{u_1^2 u_{11}} j_{11} + \frac{u_3 u_5^2}{u_1^2 u_{11}} q_{11} + \frac{u_3^3 u_5}{u_1^3 u_{11}} m_{11} + \frac{u_3^5}{u_1^4 u_{11}} n_{11}$$

The differential equatios that the functions f_1, f_3, g_5, h_5, g_7, h_7, k_7, g_9, h_9, k_9, j_9, q_9, g_{11}, h_{11}, k_{11}, j_{11}, q_{11}, m_{11}, n_{11} have to satisfy are:

$$f_1^{'2} - f_1 f_1^{''} = 1 + f_1^{'''}$$

$$3 f_1^{'} f_3^{'} - f_1 f_3^{''} - 3 f_1^{''} f_3 = 1 + f_3^{'''}$$

$$6 f_1^{'} g_5^{'} - f_1 g_5^{''} - 5 f_1^{''} g_5 = 1 + g_5^{'''}$$

$$6 f_1^{'} h_5^{'} + 8 f_3^{'2} - f_1 h_5^{''} - 5 f_1^{''} h_5 - 8 f_3 f_3^{''} = \frac{1}{2} + h_5^{'''}$$

$$8 f_1^{'} g_7^{'} - f_1 g_7^{''} - 7 f_7 f_1^{''} = 1 + g_7^{'''}$$

$$8 f_1^{'} h_7^{'} + 24 f_3^{'} g_5^{'} - f_1 h_7^{''} - 9 f_3 g_5^{''} - 15 f_3^{''} g_5 - 7 f_1^{''} h_7 = 1 + h_7^{'''}$$

$$8 f_1^{'} k_7^{'} + 24 f_3^{'} h_5^{'} - f_1 k_7^{''} - 9 f_3 h_5^{''} - 15 f_3^{''} h_5 - 7 f_1^{''} k_7 = k_7^{'''}$$

$$10 f_1^{'} g_9^{'} - f_1 g_9^{''} - 9 f_1^{''} g_9 = 1 + g_9^{'''}$$

$$10f_1^{'}h_9^{'}+32f_3^{'}g_7^{'}-f_1h_9^{''}-9{,}6f_3g_7^{''}-22{,}4f_3^{''}g_7-9f_1^{''}h_9=1+h_9^{'''}$$

$$10f_1^{'}k_9^{'}+18g_5^{'2}-f_1k_9^{''}-18g_5g_5^{''}-9f_1^{''}k_9=\frac{1}{2}+k_9^{'''}$$

$$10f_1^{'}j_9^{'}+32f_3^{'}h_7^{'}+36g_5^{'}h_5^{'}-f_1j_9^{''}-9{,}6f_3h_7^{''}-18g_5h_5^{''}-18g_5^{''}h_5-$$
$$22{,}4f_3^{''}h_7-9f_1^{''}j_9=j_9^{'''}$$

$$10f_1^{'}q_9^{'}+32f_3^{'}k_7^{'}+18h_5^{'2}-f_1q_9^{''}-9{,}6f_3k_7^{''}-18h_5h_5^{''}-22{,}4f_3^{''}k_7-$$
$$9f_1^{''}q_9=q_9^{'''}$$

$$12f_1^{'}g_{11}^{'}-f_1g_{11}^{''}-11f_1^{''}g_{11}=1+g_{11}^{'''}$$

$$12f_1^{'}h_{11}^{'}+40f_3^{'}g_9^{'}-f_1h_{11}^{''}-10f_3g_9^{''}-30f_3^{''}g_9-11f_1^{''}h_{11}=1+h_{11}^{'''}$$

$$12f_1^{'}k_{11}^{'}+48g_5g_7^{'}-f_1k_{11}^{''}-20g_5g_7^{''}-28g_5^{''}g_7-11f_1^{''}k_{11}=1+k_{11}^{'''}$$

$$12f_1^{'}j_{11}^{'}+40f_3^{'}h_9^{'}+48h_5^{'}g_7^{'}-f_1j_{11}^{''}-10f_3h_9^{''}-20h_5g_7^{''}-28h_5^{''}g_7-$$
$$30f_3^{''}h_9-11f_1^{''}j_{11}=j_{11}^{'''}$$

$$12f_1^{'}q_{11}^{'}+40f_3^{'}k_9^{'}+48g_5^{'}h_7^{'}-f_1q_{11}^{''}-10f_3k_9^{''}-20g_5h_7^{''}-28g_5^{''}h_7-$$
$$30f_3^{''}k_9-11f_1^{''}q_{11}=q_{11}^{'''}$$

$$12f_1^{'}m_{11}^{'}+40f_3^{'}j_9^{'}+48h_5^{'}k_7^{'}+48g_5^{'}h_7^{'}-f_1m_{11}^{''}-10f_3j_9^{''}-20h_5h_7^{''}-$$
$$20g_5k_7^{''}-28g_5^{''}k_7-28h_5^{''}h_7-30f_3^{''}j_9-11f_1^{''}m_{11}=m_{11}^{'''}$$

$$12f_1^{'}n_{11}^{'}+40f_3^{'}q_9^{'}-f_1n_{11}^{''}-10f_3q_9^{''}-20h_5k_7^{''}-28h_5^{''}k_7-30f_3^{''}q_9-$$
$$11f_1^{''}n_{11}=n_{11}^{'''}$$

The $u(x,y)=0$ at $y=0$ and $\lim_{y\to\infty}u(x,y)=U(x)$ boundary conditios in terms of f_1, f_3, g_5, h_5, g_7, h_7, k_7, g_9, h_9, k_9, j_9, q_9, g_{11}, h_{11}, k_{11}, j_{11}, q_{11}, m_{11} ,n_{11} become

at $\eta=0$

$$f_1=f_1^{'}=0;\quad f_3=f_3^{'}=0;\quad g_5=g_5^{'}=0;\quad h_5=h_5^{'}=0;\quad g_7=g_7^{'}=0;$$
$$h_7=h_7^{'}=0;\quad k_7=k_7^{'}=0;\quad g_9=g_9^{'}=0;\quad h_9=h_9^{'}=0;\quad k_9=k_9^{'}=0;$$

$j_9 = j_9^{'} = 0;\ q_9 = q_9^{'} = 0;\ \ g_{11} = g_{11}^{'} = 0;\ \ h_{11} = h_{11}^{'} = 0;\ \ k_{11} = k_{11}^{'} = 0;$
$j_{11} = j_{11}^{'} = 0;\ \ q_{11} = q_{11}^{'} = 0;\ \ m_{11} = m_{11}^{'} = 0;\ \ n_{11} = n_{11}^{'} = 0;$

and at $\eta = \infty$

$$f_1^{'} = 1;\ f_3^{'} = \frac{1}{4};\ g_5^{'} = \frac{1}{6};\ h_5^{'} = 0;\ g_7^{'} = \frac{1}{8};\ h_7^{'} = 0;\ k_7^{'} = 0;\ g_9^{'} = \frac{1}{10};$$
$$h_9^{'} = 0;\ k_9^{'} = 0;\ j_9^{'} = 0;\ q_9^{'} = 0;\ g_{11}^{'} = \frac{1}{12};\ h_{11}^{'} = 0;\ k_{11}^{'} = 0;\ j_{11}^{'} = 0;$$
$$q_{11}^{'} = 0;\ m_{11}^{'} = 0;\ n_{11}^{'} = 0;$$

Conclusion

We have derived the ordinary differential equations necessary to carry out numerical approximation to the solution of the boundary layer equation by 11-th order Blasius method.

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