Pseudo-analysis approach to nonlinear partial differential equations

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Abstract: An overview of methods of pseudo-analysis in applications on important classes of nonlinear partial differential equations, occurring in different fields, is given. Hamilton-Jacobi equations, specially important in the control theory, are for important models usually with non-linear Hamiltonian H which is also not smooth, e.g., the absolute value, min *or* max *operations, where it can not apply the classical mathematical analysis. Using the pseudo-analysis with generalized pseudo-convolution it is possible to obtain solutions which can be interpreted in the mentioned classical way. Another important classes of nonlinear equations, where there are applied the pseudo-analysis, are the Burgers type equations and Black and Shole equation in option pricing. Very recent applications of pseudo-analysis are obtained on equations which model fluid mechanics (Navier-Stokes equation) and image processing (Perona and Malik equation).*

Keywords: Pseudo-analysis, nonlinear partial differential equation, Hamilton-Jacobi equation, Burgers type equation, Bellman differential equation, Navier-Stokes equation, Perona and Malik equation.

1 Introduction

The pseudo-analysis, see [12, 16, 17, 20, 21, 22], is based, instead of the usual field of real numbers, on a semiring acting on the real interval $[a, b] \subset [-\infty, \infty]$, denoting the corresponding operations as $\oplus$ (pseudo-addition) and $\odot$ (pseudo-multiplication), see Section 2. It is applied, as universal mathematical theory, successfully in many fields, e.g., fuzzy systems, decision making, optimization theory, differential equations, etc. This structure is applied for solving nonlinear equations (ODE, PDE, difference equations, etc.) using the pseudo linear principle, which means that if u_1 and u_2 are solutions of the considered nonlinear equation, than also $a_1 \odot u_1 \oplus a_2 \odot u_2$ is a solution for any constants a_1 and a_2 from $[a, b]$. Based on the semiring structure (see [13]) it is developed in

[17, 18, 19, 20, 21, 22, 24] the so called pseudo-analysis in an analogous way as classical analysis, introduced $\oplus$-measure, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. There is so called "viscosity solution" method (see [14]) which gives upper and lower solutions but not a solution in the classical sense, i.e., that its substitution into the equation reduces the equation to the identity. There is given an overview of methods of pseudo-analysis in applications on important classes of nonlinear partial differential equations occurring in different fields, see [7, 8, 12, 16, 18, 19, 20, 21, 22, 24].

First we will show in Section 3 the pseudo linear superposition principle on the Burgers equation and in the limit case on a Hamilton-Jacobi equation. Pseudo-analysis was applied for finding weak solution of Hamilton-Jacobi equation with non-smooth Hamiltonian, [16, 22, 24], see Section 4. Another important class of nonlinear equations, where it is applied the pseudo-analysis, is the Black and Shole equation in option pricing, see Section 6. Very recent applications of pseudo-analysis are obtained on equations which model fluid mechanics, see Section 7. In the section 8 it is presented a general form of PDE for image restoration and there is given a connection with Gaussian linear filtering. The starting PDE in image restoration is the heat equation. Because of its oversmoothing property (edges get smeared), it is necessary to introduce some nonlinearity. Framework to study this equation is nonlinear semigroup theory ([1, 2, 4]). It is proved that Perona and Malik equation satisfy the pseudo linear superposition.

2 Pseudo-analysis

Let $[a, b]$ be closed (in some cases semiclosed) subinterval of $[-\infty, +\infty]$. We consider here a total order $\leq$ on $[a, b]$. The operation $\oplus$ (pseudo-addition) is function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is continuous, commutative, non-decreasing, associative and has a zero element, denoted by $\mathbf{0}$. Let $[a, b]_+ = \{x : x \in [a, b], x \geq \mathbf{0}\}$. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is continuous, commutative, positively non-decreasing, i.e., $x \leq y$ implies $x \odot z \leq y \odot z, z \in [a, b]_+$,associative and for which there exist a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], 1 \odot x = x$. We suppose $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

The structure $([a, b], \oplus, \odot)$ is called a *semiring* (see [13, 20]). We consider here two special important cases $([0, \infty), \min, +)$ and the g-calculus, i.e., there exists a bijection $g : [a, b] \to [0, \infty]$ such that $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$.

There is introduced $\oplus$-measure $m : \mathcal{A} \to [a, b]$ on a σ-algebra $\mathcal{A}$ of subsets of a given set X, and the corresponding pseudo-integral, see [20]. Important cases are $([0, \infty), \min, +)$ and g-calculus, where the corresponding integrals are given, for $m_\varphi(A) = \inf_x \varphi(x)$ by

$$\int^{\min} f(x)\,dx = \inf_x (f(x) + \varphi(x)),$$

and by

$$\int^{g} f(x)\,dx = g^{-1}\left(\int g(f(x))\,dx\right),$$

respectively.

The *pseudo-character* of group $(G, +)$, $G \subset \mathbb{R}^n$, is a continuous (with respect to the usual topology of reals) map $\xi : G \to [a, b]$, of the group $(G, +)$ into the semiring $([a, b], \oplus, \odot)$, with the property

$$\xi(x + y) = \xi(x) \odot \xi(y), \quad x, y \in G.$$

The map $\xi \equiv \mathbf{0}$ is the trivial pseudo-character. The forms of the pseudo-character in the special cases can be found in [9, 24], where for important cases $([0, \infty), \max, +)$ and g-calculus we have $\xi(x, c) = c \cdot x$ and $\xi(x, c) = g^{-1}(e^{cx})$, respectively, for each $c \in \mathbb{R}$.

Definition 2.1 *The* pseudo-Laplace transform $\mathcal{L}^\oplus(f)$ *of a function* $f \in B(G, [a, b])$ *is defined by*

$$(\mathcal{L}^\oplus f)(\xi)(z) = \int^{\oplus}_{G\cap[0,\infty)^n} \xi(x, -z) \odot \, dm_f(x),$$

where ξ *is the pseudo-character.*

When at least pseudo-addition is idempotent operation we can consider the second type of pseudo-Laplace transform:

$$(\mathcal{L}^\oplus f)(\xi)(z) = \int^{\oplus}_{G} \xi(x, -z) \odot \, dm_f(x),$$

i.e., pseudo-integral has been taken over the whole G.

For the special important cases $([0, \infty), \max, +)$ and g-calculus, we have that the pseudo-Laplace transform has the following form

$$(\mathcal{L}^{\min} f)(z) = \inf_x (-xz + f(x)),$$

and

$$(\mathcal{L}^g f)(z) = g^{-1}\left(\int_0^\infty e^{-xz} g(f(x))\,dx\right),$$

respectively.

3 Two simpe examples of nonlinear PDE

We start with two examples to illustrate how can be applied the pseudo-linear superposition principle on some non-linear partial differential equations.

An important nonlinear partial differential equation is the Burgers equation for a function $u = u(x,t)$. Burgers (1948), Hopf (1950) and Cole (1951) investigated as a model of turbulence the following equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{c}{2}\frac{\partial^2 v}{\partial x^2}, \tag{1}$$

where c is a parameter. Putting $v = \frac{\partial u}{\partial x}$ in (1) and integrating with respect to x we obtain the equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{c}{2}\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

for $x \in \mathbb{R}$ and $t > 0$, with the initial condition $u(x,0) = u_0(x)$, where c is the given positive constant, and which models the burning of a gas in a rocket. We shall apply on this equation the g-calculus, with the generator $g(u) = e^{-u/c}$. Then, the corresponding pseudo-addition is $u \oplus v = -c\ln(e^{-u/c} + e^{-v/c})$, and the distributive pseudo-multiplication $u \odot v = u + v$. Then for solutions u_1 and u_2 of (2) the function $(\lambda_1 \odot u_1) \oplus (\lambda_2 \odot u_2)$ is also a solution of Burgers equation (2). The solution of the given initial problem is

$$u(x,t) = \frac{c}{2}\ln(2\pi ct) \odot \int^{\oplus} \frac{(x-s)^2}{2t} \odot u_0(s)\, ds.$$

Taking $c \to 0$ in the Burgers equation (2) we obtain Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0.$$

Then for solutions u_1 and u_2 the function $(\lambda_1 \odot u_1) \oplus (\lambda_2 \odot u_2)$, where

$$u \oplus v = \min(u,v) \text{ and } u \odot v = u + v,$$

is also a solution of the preceding Hamilton-Jacobi equation.

4 Hamilton-Jacobi equation with non-smooth Hamiltonian

We consider here the nonlinear PDE, so called Hamilton-Jacobi-Bellman equation

$$\frac{\partial u(x,t)}{\partial t} + H\left(\frac{\partial u}{\partial x}, x, t\right) = 0, \tag{3}$$

see [12, 16, 20, 21, 22, 24]. Hamilton-Jacobi equations are specially important in the control theory. Unfortunately, usually the interesting models are represented by Hamilton-Jacobi equations in which the non-linear Hamiltonian H is not smooth, for example the absolute value, min or max operations. Hence we can not apply on such cases the classical mathematical analysis. There is so called "viscosity solution" method (see [14]) which gives upper and lower solutions but not a solution in the classical sense, i.e., that its substitution into the equation reduces the equation to the identity. Using the pseudo-analysis with generalized pseudo-convolution it is possible to obtain solutions which can be interpreted in the mentioned classical way.

We extend now the pseudo-superposition principle to a more general case, see [12, 21, 22].

Theorem 4.1 *If u_1 and u_2 are solutions of the Hamilton-Jacobi equation (3), where $H \in C(\mathbb{R}^{n+2})$ and $\frac{\partial u}{\partial x}$ is the gradient of u, then $(\lambda_1 \odot u_1) \oplus (\lambda_2 \odot u_2)$ is also a solution of the Hamilton-Jacobi equation (3), with respect to the operations $\oplus = \min$ and $\odot = +$.*

Let $C_{\mathbf{0}}(\mathbb{R}^n)$ be the space of continuous functions $f : \mathbb{R}^n \to P$ (P is of type $(\min, +)$ or $(\min, \max)$) with the property that for each $\varepsilon > 0$ there exists a compact subset $K \subset \mathbb{R}^n$ such that $d(\mathbf{0}, \inf_{x \in \mathbb{R}^n \setminus K} f(x)) < \varepsilon$, with the metric $D(f, g) = \sup_x d(f(x), g(x))$. Let $C_{\mathbf{0}}^{cs}(\mathbb{R}^n)$ be the subspace of $C_{\mathbf{0}}(\mathbb{R}^n)$ of functions f with compact support $\mathrm{supp}_{\mathbf{0}} = \overline{\{x | \ f(x) \neq \mathbf{0}\}}$. The dual semimodul $(C_{\mathbf{0}}(\mathbb{R}^n))^*$ is the semimodul of continuous pseudo-linear P-valued functionals on $C_{\mathbf{0}}(\mathbb{R}^n)$ (with respect to pointwise operations). Analogously the dual semimodul $(C_{\mathbf{0}}^{cs}(\mathbb{R}^n))^*$ is the semimodul of continuous pseudo-linear P-valued functionals on $C_{\mathbf{0}}^{cs}(\mathbb{R}^n)$ (with respect to pointwise operations). We shall need the following representation theorem, see [12].

Theorem 4.2 *Let f be a function defined on $\mathbb{R}^n$ and with values in the semiring P of type $(\min, +)$ or $(\min, \max)$, and a functional $m_f : C_{\mathbf{0}}^{cs}(\mathbb{R}^n) \to P$ is given by*

$$m_f(h) = \int^{\oplus} f \odot dm_h = \inf_x (f(x) \odot h(x)).$$

Then

1) *The mapping $f \mapsto m_f$ is a pseudo-isomorphism of the semimodule of lower semicontinuous functions onto the semimodule $(C_{\mathbf{0}}^{cs}(\mathbb{R}^n))^*$.*

2) *The space $C_{\mathbf{0}}^*(\mathbb{R}^n)$ is isometrically isomorphic with the space of bounded functions, i.e., for every $m_{f_1}, m_{f_2} \in C_{\mathbf{0}}^*(\mathbb{R}^n)$ we have*

$$\sup_x d(f_1(x), f_2(x))$$

$$= \sup\{d(m_{f_1}(h), m_{f_2}(h)) : h \in C_{\mathbf{0}}(\mathbb{R}^n), D(h, \mathbf{0}) \leq 1\}.$$

3) *The functionals m_{f_1} and m_{f_2} are equal if and only if $Clf_1 = Clf_2$, where*

$$Clf(x) = \sup\{\psi(x) : \psi \in C(\mathbb{R}^n), \psi \leq f\}.$$

We consider now the following *Cauchy problem for Hamilton-Jacobi(- Bellman) equation*

$$\frac{\partial u}{\partial t} + H\left(\frac{\partial u}{\partial x}\right) = 0, \quad u(x,0) = u_0(x), \tag{4}$$

where $x \in \mathbb{R}^n$, and the function $H : \mathbb{R}^n \to \mathbb{R}$ is convex (by boundedness of H it is also continuous). For control theory the important examples of the Hamiltonian H are non-smooth functions, e.g., max and $|.|$. The approach with pseudo-analysis avoids the use of the so called "viscosity solution" method, which does not give the exact solution of (4) (see [14]). We apply now the methods of pseudo-analysis. For that purpose we define the family of operators $\{R_t\}_{t>0}$, for a function $u_0(x)$ bounded from below in the following way

$$u(t,x) = (R_t u_0)(x) = \inf_{z \in \mathbb{R}^n} (u_0(z) - t\mathcal{L}^{\min}(H)(\frac{x-z}{t})), \tag{5}$$

where $\mathcal{L}$ is considered on the whole $\mathbb{R}^n$. The operator R_t is pseudo-linear with respect to $\oplus = \min$ and $\odot = +$, where $\mathcal{L}^{\oplus}(H)(q) = \inf_{p \in \mathbb{R}^n} (-pq + H(p))$.

First we suppose that u_0 is smooth and strongly convex. We shall use the notations $< x, y >$ and $\|x\|$ for the scalar product and Euclidean norm in $\mathbb{R}^n$, respectively. For a function $F : \mathbb{R}^n \to [-\infty, +\infty]$ its *subgradient* at a point $u \in \mathbb{R}^n$ is a point $w \in \mathbb{R}^n$ such that $F(u)$ is finite and

$$< w, v-u > + F(u) \leq F(v)$$

for all $v \in \mathbb{R}^n$. Then we have by [12].

Lemma 4.3 *Let $u_0(x)$ be smooth and strongly convex and there exists $\delta > 0$ such that for all x the eigenvalues of the matrix $u_0''(x)$ of all second derivatives are not less than δ. Then*

1) *For every $x \in \mathbb{R}^n$, $t > 0$,there exists a unique $\xi(t,x) \in \mathbb{R}^n$ such that $\frac{x-\xi(t,x)}{t}$ is a subgradient of the function H at the point $u_0'(\xi(t,x))$ and*

$$(R_t u_0)(x) = u_0(\xi(t,x)) - t\mathcal{L}^{\min}(H)(\frac{x-\xi(t,x)}{t}).$$

2) *The function $\xi(t,x)$ for $t > 0$ satisfies the Lipschitz condition on compact sets, and $\lim_{t\to 0} \xi(t,x) = x$.*

3) *The Cauchy problem (4) has a unique C^1 solution given by (4.3), and*

$$\frac{\partial u}{\partial x}(t,x) = u_0'(\xi(t,x)).$$

The Cauchy problem

$$\frac{\partial u}{\partial t} + H\left(-\frac{\partial u}{\partial x}\right) = 0, \qquad (6)$$
$$u(0,x) = u_0(x),$$

is the adjoint problem of the Cauchy problem (4). The classical resolving operator R_t^* of the Cauchy problem (6) on the smooth convex functions by Lemma 4.3 is given by

$$(R_t^* u_0)(x) = \inf_{\xi}(u_0(\xi) - t\mathcal{L}^{\min}(H)(\frac{\xi - x}{t})).$$

We note that R_t^* is the *adjoint of the resolving operator* R_t with respect to bipseudo-linear functional

$$\int_{\mathbb{R}^n}^{\oplus} f \odot h \, dm.$$

Then we can introduce, as in the theory of linear equation, the notion of generalized weak solution (using Theorem 4.2), see [12].

Definition 4.4 *Let u_0 be a bounded from below function $u_0 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and m_{u_0} the corresponding functional from $C_{\mathbf{0}}^*(\mathbb{R}^n)$. The generalized weak pseudo solution of Cauchy problem (4) is a continuous function from below $(R_t u_0)(x)$ which is defined uniquely by*

$$m_{R_t u_0}(\varphi) = m_{u_0}(R_t^* \varphi)$$

for all smooth convex functions φ.

We can construct the solution for the case when u_0 is a smooth strictly convex function by Lemma 4.3. Then it follows by Theorem 4.2 and Definition 4.4.

Theorem 4.5 *For an arbitrary function $u_0(x)$ bounded from below the weak pseudo-solution of the Cauchy problem (4) is given by*

$$(R_t u_0)(x) = (R_t Clu_0)(x) = \inf_{z}(Clu_0(z) + t\mathcal{L}^{\min}(H)(\frac{x - z}{t})),$$

where

$$Clf(x) = \sup\{\psi(x) : \psi \in C(\mathbb{R}^n), \psi \leq f\}.$$

5 Bellman differential equation for multicriteria optimization problems

We present results from [12] obtained for the controlled process in $\mathbb{R}^n$ specified by a controlled differential equation $\dot{x} = f(x, v)$ (where v belongs to a metric control space V) and by a continuous function $\varphi \in B(\mathbb{R}^n \times V, \mathbb{R}^k)$, which determines a vector-valued integral criterion

$$\Phi(x(\,\cdot\,)) = \int_0^t \varphi(x(\tau), u(\tau))\, d\tau$$

on the trajectories. Let us pose the problem of finding the Pareto set $\omega_t(x)$ for a process of duration t issuing from x with terminal set determined by some function $\omega_0 \in B(\mathbb{R}^n, \mathbb{R}^k)$, that is,

$$\omega_t(x) = \mathrm{Min} \bigcup_{x(\cdot)} (\Phi(x(\,\cdot\,)) \odot \omega_0(x(t))), \tag{7}$$

where $x(\,\cdot\,)$ ranges over all admissible trajectories issuing from x. We can encode the functions $\omega_t \in B(\mathbb{R}^n, P\mathbb{R}^k)$ by the functions

$$u(t, x, a)\colon \mathbb{R}_+ \times \mathbb{R}^n \times L \to \mathbb{R}.$$

The optimality principle permits us to write out the following equation, which is valid modulo $O(\tau^2)$ for small τ:

$$u(t, x, a) = \mathrm{Min}_v (h_{\tau\varphi(x,v)} \star u(t-\tau, x + \Delta x(v)))(a).$$

It follows from the representation of $h_{\tau\varphi(x,v)}$ and from the fact that n is, by definition, the multiplicative unit in $CS_n(L)$ that

$$u(t, x, a) = \min_v (\tau\overline{\varphi}(x, v) + u(t-\tau, x + \Delta x(v), a - \tau\varphi_L(x, v))).$$

Let us substitute $\Delta x = \tau f(x, v)$ into this equation, expand S in a series modulo $O(\tau^2)$, and collect similar terms. Then we obtain the equation

$$\frac{\partial u}{\partial t} + \max_v \left(\varphi_L(x, v) \frac{\partial u}{\partial a} - f(x, v) \frac{\partial u}{\partial x} - \overline{\varphi}(x, v) \right) = 0. \tag{8}$$

Although the presence of a vector criterion has resulted in a larger dimension, this equation coincides in form with the usual Bellman differential equation. Consequently, the generalized solutions can be defined on the basis of the idempotent superposition principle, as Section 4. We have the following result by [12].

Theorem 5.1 *The Pareto set $\omega_t(x)$ (7) is determined by a generalized solution $u_t \in B(\mathbb{R}^n, CS_n(L))$ of (8) with the initial condition $u_0(x) = h_{\omega_0(x)} \in B(\mathbb{R}^n, CS_n(L))$. The mapping $R_{CS}\colon u_0 \mapsto u_t$ is a linear operator on $B(\mathbb{R}^n, CS_n(L))$.*

6 Option pricing

Black-Sholes and Cox-Ross-Rubinstein formulas are basic results in the modern theory of option pricing in financial mathematics. They are usually deduced by means of stochastic analysis; various generalizations of these formulas were proposed using more sophisticated stochastic models for common stocks pricing evolution. The systematic deterministic approach to the option pricing leads to a different type of generalizations of Black-Sholes and Cox-Ross-Rubinstein formulas characterized by more rough assumptions on common stocks evolution (which are therefore easier to verify). This approach reduces the analysis of the option pricing to the study of certain homogeneous nonexpansive maps, which however, unlike the situations described in previous subsections, are "strongly" infinite dimensional: they act on the spaces of functions defined on sets, which are not (even locally) compact.

In the paper of [11] it was shown what type of generalizations of the standard Cox-Ross-Rubinstein and Black-Sholes formulas can be obtained using the deterministic (actually game-theoretic) approach to option pricing and what class of homogeneous nonexpansive maps appear in these formulas, considering first a simplest model of financial market with only two securities in discrete time, then its generalization to the case of several common stocks, and then the continuous limit. One of the objective was to show that the infinite dimensional generalization of the theory of homogeneous nonexpansive maps (which does not exists at the moment) would have direct applications to the analysis of derivative securities pricing. On the other hand, this approach, which uses neither martingales nor stochastic equations, makes the whole apparatus of the standard game theory appropriate for the study of option pricing.

7 Navier-Stokes and Stokes equations

Pseudo liner superposition principle was applied also on important equations of fluid mechanics [27]. We consider an incompressible homogeneous viscous flow: that means that div $\mathbf{u} = 0$, for the density $\rho = 1$, ν is the coefficient of viscosity, for the forces $\mathrm{f} = 0$. The equations of motion of this flow are the *Navier-Stokes equations*, see [6]:

$$\rho \frac{D\mathbf{u}}{Dt} = - \text{ grad p} + \nu\Delta\mathbf{u}$$

$$\text{div } \mathbf{u} = 0$$

$$\mathbf{u} = 0 \quad \text{on } \partial D$$

where $\Delta\mathbf{u}$ is the Laplacian of the velocity u, defined in this way: $\Delta\mathbf{u} = (\partial_{xx} + \partial_{yy})\mathbf{u} = (\partial_{xx}u + \partial_{yy}v)$, as $\mathbf{u}(\mathbf{x}, t) = (u(x, y, t), v(x, y, t))$.

We consider two-dimensional incompressible flow in the upper half plane $y > 0$; so the projections of the Navier-Stokes equations on axes x and y are the following:

$$\partial_t u + u\partial_x u + v\partial_y u + \partial_x p + \nu(\partial_{xx}u + \partial_{yy}u) = 0 \tag{9}$$

$$\partial_t v + u\partial_x v + v\partial_y v + \partial_y p + \nu(\partial_{xx}v + \partial_{yy}v) = 0 \tag{10}$$

$$\partial_x u + \partial_y v = 0 \tag{11}$$

$$u = v = 0 \qquad \text{on} \qquad \partial D. \tag{12}$$

We have proved in [27] the following two theorems.

Theorem 7.1 *Let $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ be two solutions of (9) - (12) and a_1, a_2 two real numbers. Then the pseudo-linear combination*

$$(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p}) = \mathbf{min}(\mathbf{max}(a_1, \mathbf{s}_{1,p)}, \mathbf{max}(a_2, \mathbf{s}_{2,p}))$$

is again a solution of (9) - (12).

Theorem 7.2 *Let $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ be two solutions of (9) - (12) which satisfy*

$$\partial_y u_i = \partial_y v_i \qquad i = 1, 2.$$

Then the pseudo-linear combination $(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p})$, for two real numbers a_1, a_2 where $\odot$ is given by

$$\lambda \odot \mathbf{s} = \lambda \odot (u, v, p) = (\lambda + u, \lambda + v, \lambda + p),$$

is again a solution of (9) - (12).

The *Stokes equations* approximate equations for incompressible flow ([5]):

$$\partial_t \mathbf{u} + \text{ grad p} + \nu \Delta \mathbf{u} = 0 \tag{13}$$

$$\text{div } \mathbf{u} = 0 \tag{14}$$

We have proved in [27] the following theorem.

Theorem 7.3 *Let $\mathbf{s}_i(t) = (u_i(t), v_i(t), p_i(t))$, $i = 1, 2$ be solutions of (13) and (14). Then the pseudo-linear combination $(a_1 \odot \mathbf{s}_1) \oplus (a_2 \odot \mathbf{s}_2)$, for two real numbers a_1, a_2 where $\oplus$ is given by $(\mathbf{s}_1 \oplus \mathbf{s}_2)$*

$$= (g^{-1}(g(u_1) + g(u_2), g^{-1}(g(v_1) + (g(v_2), g^{-1}(g(p_1) + g(p_2)),$$

and

$$a \odot \mathbf{s} = ((g^{-1}(g(a) \cdot g(u), g^{-1}(g(a) \cdot g(v), (g^{-1}(g(a) \cdot g(p))$$
$$= (a + u, a + v, a + p)$$

with g defined by $g(a) = e^{-\ c\ a}, c > 0$ and $g^{-1}(b) = -\frac{1}{c}\ \log\ b$, is again solution of (13) - (14).

8 Pseudo-linear superposition principle for Perona and Malik equation

Partial differential equations are applied for image processing ([1, 3, 28]). In that method a restored image can be seen as a version of the initial image at a special scale. An image u is embedded in an evolution process, denoted by $u(t,\cdot)$. The original image is taken at time $t=0$, $u(0,\cdot)=u_0(\cdot)$. The original image is then transformed, and this process can be written in the form $\frac{\partial u}{\partial t}(t,x)+F\left(x,u(t,x),\nabla u(t,x),\nabla^2 u(t,x)\right)=0$ in Ω. Some possibilities for F to restore an image are considered in [1]. PDE-methods for restoration is in general form:

$$\left\{\begin{array}{l} \frac{\partial u}{\partial t}(t,x)+F\left(x,u(t,x),\nabla u(t,x),\nabla^2 u(t,x)\right)=0 \text{ in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial N}(t,x)=0 \text{ on } (0,T)\times\partial\Omega, \quad u(0,x)=u_0(x), \end{array}\right. \tag{15}$$

where $u(t,x)$ is the restored version of the initial degraded image $u_0(x)$. The idea is to construct a family of functions $\{u(t,x)\}_{t>0}$ representing successive versions of $u_0(x)$. As t increases $u(t,x)$ changes into a more and more simplified image. We would like to attain two goals. The first is that $u(t,x)$ should represent a smooth version of $u_0(x)$, where the noise has been removed. The second, is to be able to preserve some features such as edges, corners, which may be viewed as singularitis. The basic PDE in image restoration is the heat equation:

$$\left\{\begin{array}{c} \frac{\partial u}{\partial t}(t,x)-\Delta u(t,x)=0, t\geq 0, x\in\mathbb{R}^2, \\ u(0,x)=u_0(x). \end{array}\right. \tag{16}$$

We consider that $u_0(x)$ is primarily defined on the square $[0,1]^2$. We extend it by symmetry to $C=[-1,1]^2$, and then on all $\mathbb{R}^2$, by periodicity. This way of extending $u_0(x)$ is classical in image processing. If $u_0(x)$ is extended in this way and satisfies in addition $\int_C |u_0(x)|\,dx<+\infty$, we will say that $u_0\in L^1_{\#}(C)$ (see [1]). Solving (16) is equivalent to carrying out a Gaussian linear filtering, which was widely used in signal processing. If $u_0\in L^1_{\#}(C)$, then the explicit solution of (16) is given by

$$u(t,x)=\int_{\mathbb{R}^2} G_{\sqrt{2t}}(x-y)\,u_0(y)\,dy=\left(G_{\sqrt{2t}}*u_0\right)(x),$$

where $G_\sigma(x)$ denotes the two-dimensional Gaussian kernel

$$G_\sigma(x)=\frac{1}{2\pi\sigma}e^{-\frac{|x|^2}{2\sigma^2}}$$

The heat equation has been (and is) successfully applied in image processing but it has some drawback. It is too smoothing and because of that edges can be lost or severely blurred. In [1] authors consider models that are generalizations of the heat equation. The domain image will be a bounded open set Ω of $\mathbb{R}^2$.

The following equation is initially proposed by Perona and Malik [28]:

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}\left(c\left(|\nabla u|^2\right)\nabla u\right) \quad \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial N} = 0 \text{ on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega \end{cases} \tag{17}$$

where $c : [0,\infty) \to (0,\infty)$. If we choose $c \equiv 1$, then it is reduced on the heat equation. If we assume that $c(s)$ is a decreasing function satisfying $c(0) = 1$ and $\lim_{s\to\infty} c(s) = 0$, then inside the regions where the magnitude of the gradient of u is weak, equation (17) acts like the heat equation and the edges are preserved. For each point x where $|\nabla u| \neq 0$ we can define the vectors $N = \frac{\nabla u}{|\nabla u|}$ and T with $T \cdot N = 0$, $|T| = 1$. For the first and second partial derivatives of u we use the usual notation u_{x_1}, u_{x_2}, $u_{x_1x_1}$,... We denote by u_{NN} and u_{TT} the second derivatives of u in the T-direction and N-direction, respectively:

$$\begin{aligned} u_{TT} &= T^t\nabla^2 uT = \frac{1}{|\nabla u|^2}\left(u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy}\right), \\ u_{NN} &= N^t\nabla^2 uN = \frac{1}{|\nabla u|^2}(u_x^2 u_{xx} + u_y^2 u_{yy} + 2u_x u_y u_{xy}). \end{aligned}$$

The first equation in (17) can be written as

$$\frac{\partial u}{\partial t}(t,x) = c\left(|\nabla u(t,x)|^2\right) u_{TT} + b\left(|\nabla u(t,x)|^2\right) u_{NN}, \tag{18}$$

where $b(s) = c(s) + 2sc'(s)$. Therefore, (18) is a sum of a diffusion in the T-direction and a diffusion in the N-direction. The function c and b act as weighting coefficients. Since N is normal to the edges, it would be preferable to smooth more in the tangential direction T than in the normal direction. Because of that we impose

$$\lim_{s\to\infty}\frac{b(s)}{c(s)} = 0 \text{ or } \lim_{s\to\infty}\frac{sc'(s)}{c(s)} = -\frac{1}{2} \tag{19}$$

If $c(s) > 0$ with power growth, then (19) implies that $c(s) \approx 1/\sqrt{s}$ as $s \to \infty$. The equation (17) is parabolic if $b(s) > 0$. The assumptions imposed on $c(s)$ are

$$\begin{cases} c : [0,\infty) \to (0,\infty) \text{ decreasing}, \\ c(0) = 1,\ c(s) \approx \frac{1}{\sqrt{s}} \text{ as } s \to \infty, \\ b(s) = c(s) + 2sc'(s) > 0. \end{cases} \tag{20}$$

Often used function $c(s)$ satisfying (20) is $c(s) = \frac{1}{\sqrt{1+s}}$. Because of the behavior $c(s) \approx 1/\sqrt{s}$ as $s \to \infty$, it is not possible to apply general results from parabolic equations theory. Framework to study this equation is nonlinear semigroup theory (see [1, 2, 4]).

We have proved in [25] that the pseudo-linear superposition principle holds for Perona and Malik equation.

Theorem 8.1 *If* $u_1 = u_1(t,x)$ *and* $u_2 = u_2(t,x)$ *are solutions of the equation*

$$\frac{\partial u}{\partial t} - div\left(c\left(|\nabla u|^2\right)\nabla u\right) = 0, \tag{21}$$

then $u_1 \oplus u_2$ *is also a solution of (21) on the set*

$$D = \{(t,x)\,|t \in (0,T), x \in \mathbb{R}^2, u_1(t,x) \neq u_2(t,x)\},$$

with respect to the operation $\oplus = \min$.

The obtained results will serve for further investigation of the weak solutions of the equation (21) in the sense of Maslov [10, 12, 22, 23] and Gondran [7, 8], as well as their important applications.

9 Conclusion

The pseudo-linear superposition principle, as it was shown, allows us to transfer the methods of linear equations to many important nonlinear partial differential equations. Some further developments related more general pseudo-operations with applications on nonlinear partial differential equations were obtain in [22, 23, 26].

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