# An Axiomatizatin of Qualitative Probability

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*Abstract: Qualitative reasoning attracts special attention over the last two decades due to its wide applicability in every-day tasks such as diagnostics, tutoring, real-time monitoring, hazard identification, etc. Reasoning about qualitative probabilities is one of the most common cases of qualitative reasoning. Here we will present a part of our work on the problem of sound, strongly complete and decidable axiomatization of the notion of qualitative probability.*

## 1 Introduction

Reasoning about qualitative probabilities is one of the most prominent cases of the qualitative reasoning. Some varieties of qualitative probability are discussed in [18]. Here we will present a part of our work on the axiomatization of the notion of qualitative probability within the framework of probabilistic logic. Though they are infinitary, our logics are sound, strongly complete and decidable.

The standard approach to probabilistic logic [3], [9] (see also the database [10]) involves extension of the classical propositional or predicate calculus with the modal-like operators, in our notation $P(\alpha) \geq s$, with the attended meaning 'the probability of $\alpha$ is at least $s$ ', where $s$ ranges over the predefined index set $S$ . The corresponding semantics is defined as special kind of Kripke models with probability measures on worlds.

As it is well known (see [9], [17]), the key issue for this kind of logics is the non-compactness phenomena: there are examples of finitely satisfiable inconsistent sets of formulas. There are several ways to overcome this problem. In [3] a finitary axiomatization which involves the reasoning about linear inequalities was provided. However, only simple completeness (every consistent formula is satisfiable, in contrast to the strong completeness: every consistent set of formulas is satisfiable) can be proved for that logic. As a consequence, there are examples of consistent unsatisfiable sets of formulas.

In [8], [9], [11], [12], [14], [15] some infinitary probabilistic logics were presented and the corresponding strong completeness theorems were proved. In those logics we keep formulas finite and allow countably infinite inference rules (conclusions might have countably many premises). In [9], [13] and [17], some probabilistic logics with a fixed finite range for probability measures were given.

Lately, probabilistic logics with the non-Archimedean measures were introduced. For instance, in [15] was introduced a non-Archimedean probabilistic formalism that can be used for the modeling of default reasoning.

In the presence of the probabilistic operators $P(\alpha) \geq s$ one can semantically express the notion of the qualitative probability in the following way: a formula $\beta$ is at least probable as a formula $\alpha$ iff $P(\beta) \geq s$ implies $P(\alpha) \geq s$ for all $s \in S$. Building on our previous work, we have extended the probability language with an additional binary operator $\alpha \prec \beta$ with the intended meaning '$\beta$ is at least probable as $\alpha$'. Depending on the choice of the index set $S$ and the range of the models, we have developed several formal systems (see [12]). Here we will only present the basic ideas.

# 2 Syntax and Semantics

By $Var$ we will denote the set of propositional variables. We assume that there are countably many propositional variables. The corresponding set of propositional formulas will be denoted by $For_C$. Propositional formulas will be denoted by $\alpha$, $\beta$ and $\gamma$, indexed if necessary. The index set $S$ is defined as the set of all rational numbers in the real unit interval $[0,1]$. The elements of $S$ will be denoted by $r$ and $s$, indexed if necessary.

A basic probabilistic formula is any formula of the following two forms:

- $P(\alpha) \geq s$;
- $\alpha \prec \beta$.

Abbreviations such as $P(\alpha) > s$, $P(\alpha) \leq s$, etc. are defined as usual (see [12]). The set $For_P$ of all probabilistic formulas is the Boolean closure of the basic probabilistic formulas. Probabilistic formulas will be denoted by $\phi$, $\psi$ and $\theta$, indexed if necessary.

A model is any structure $M = \langle W, H, \mu, v \rangle$ with the following properties:

- $W$ is a nonempty set.

- $H$ is an algebra of sets on $W$.
- $\mu : H \to [0,1]$ is a finitely additive probability measure.
- $v : For_C \times W \to \{0,1\}$ is a truth assignment.

For $\alpha \in For_C$, by $[\alpha]$ we will denote the set of all $w \in W$ such that $v(\alpha, w) = 1$. A model $M$ is measurable if $[\alpha] \in H$ for all $\alpha \in For_C$.

Let $M = \langle W, H, \mu, v \rangle$ be a measurable model. The satisfiability relation is defined recursively as follows:

- $M$ satisfies $\alpha$ if $[\alpha] = W$.
- $M$ satisfies $P(\alpha) \geq s$ if $\mu([\alpha]) \geq s$.
- $M$ satisfies $\alpha \prec \beta$ if $\mu([\alpha]) \leq \mu([\beta])$.
- $M$ satisfies $\phi \wedge \psi$ if $M$ satisfies $\phi$ and $M$ satisfies $\psi$.
- $M$ satisfies $\neg\phi$ if $M$ doesn't satisfy $\psi$.

A formula is satisfiable if there is a measurable model that satisfies it. A formula is valid if it is satisfied in every measurable model. A set of formulas is satisfiable if there is a measurable model that satisfies every formula from the set.

# 3 Axiomatization

In [12] we have shown that the following axioms and inference rules give a strongly complete characterization of valid probabilistic formulas.

**Axioms**

1 Substitutional instances of tautologies.

2 $P(\alpha) \geq 0$.

3 $P(\alpha) \geq r \to P(\alpha) > s$, whenever $r > s$.

4 $P(\alpha) > s \to P(\alpha) \geq s$.

5 $P(\alpha) \geq s \leftrightarrow P(\beta) \geq s$, whenever $\alpha \leftrightarrow \beta$ is a tautology.

6 $(P(\alpha) \geq r \wedge P(\beta) \geq s \wedge P(\alpha \wedge \beta) = 0) \to P(\alpha \vee \beta) \geq \min(1, r + s)$.

7 $(P(\alpha) \leq s \wedge P(\beta) \geq s) \rightarrow \alpha \prec \beta$.

8 $(\alpha \leq \beta \wedge P(\alpha) \geq s) \rightarrow P(\beta) \geq s$.

**Inference Rules**

1 Modus ponens for propositional formulas and modus ponens for probabilistic formulas.

2 Necessitation: from $\alpha$ derive $P(\alpha) = 1$.

3 Archimedean rule: from the set of premises $\{\phi \rightarrow P(\alpha) \geq s - n^{-1} : n > s^{-1}\}$ infer $\phi \rightarrow P(\alpha) \geq s$.

4 $\prec$ rule: from the set of premises $\{\phi \rightarrow (P(\alpha) \geq s \rightarrow P(\beta) \geq s) : s \in S\}$ infer $\phi \rightarrow \alpha \prec \beta$.

Let us briefly comment axioms and inference rules. The first axiom is necessary since all tautology instances are valid formulas (see the last two items in the definition of the satisfiability). The 2nd axiom and the necessitation rule provide that the $P$ - value of each propositional formula is between $0$ and $1$. The 3rd and 4th axiom provide the usual properties of $\geq$. The 5th axiom provides that the equivalent formulas have the same $P$ - values. The 6th axiom provides the finite additivity. The last two axioms and the $\prec$ rule axiomatize qualitative probability. Finally, the Archimedean rule provides the strong completeness of our system.

# 4 Decidability

An immediate consequence of the proposed axiomatization is the fact that each probabilistic formula has a disjunctive normal form, i.e., it is equivalent to a finite disjunction of literals, where a literal is either a basic probabilistic formula or its negation.

Thus, the question of satisfiability of probabilistic formulas is reduced to the question of satisfiability of finite conjunctions of literals. Using the standard technique (see [8]), we can equivalently reduce satisfiability of probabilistic formula to the existence of a solution of the adjoined system of linear inequalities. It is well known that the later problem is decidable.

**Conclusion**

The paper presents a probabilistic logic in which the notion of the qualitative probabiliry is completely axiomatized. Our logic involve infinitary rules in order to achieve the strong completeness, which is impossible in the finitary setting (assuming the real valued semantics and the infinite number of propositional variables). Detailed exposition with all proofs can be found in [12].

We are aware of only a few papers which present a syntactical approach to qualitative probability. An early result on the first order axiomatization of qualitative probability is due to Scott [16]. Some variants of the first order approach in the infinite setting were discussed in [6]. Qualitative probabilities are expressible in the systems introduced in [3]. However, those logics are only simply complete (finitary axiomatization and the real valued semantics). The paper [5] also provides a simply complete axiomatization of qualitative probability.

In [2], [7], [9] and [11], nesting of probabilistic operators is allowed and higher order probabilities are expressible. Our methodology can be easily extended to those cases as well.

**References**

[1] D. Dubois, H. Prade. Qualitative Possibility Functions and Integrals. In: E. Pap (Ed.), Handbook of Measure Theory, North-Holland, 1499-1522, 2002

[2] R. Fagin, J. Halpern. Reasoning about Knowledge and Probability. Journal of the ACM 41 (2), 340-367, 1994

[3] R. Fagin, J. Y. Halpern, N. Meggido. A Logic for Reasoning about Probabilities. Information and Computation 87 (1/2), 78-128, 1990

[4] D. Lehmann. Generalized Qualitative Probability: Savage Revisited. Proc. of 12th Conference on Uncertainty in Artificial Intelligence (UAI-96), E. Horvitz and F. Jensen (Eds.), 381-388, 1996

[5] E. Marchioni, L. Godo. A Logic for Reasoning about Coherent Conditional Probability: a Modal Fuzzy Logic Approach. In J. Leite and J. Alferes (Eds.), 9th European Conference Jelia'04, lecture notes in artificial intelligence (LNCS/LNAI), 3229, 213-225, 2004

[6] L. Narens. On Qualitative Axiomatizations for Probability Theory. Journal of Philosophical Logic, Vol. 9, No. 2, 143-151, Springer 1980

[7] Z. Ognjanović, M. Rašković. A Logic with Higher Order Probabilities. Publications de l'institute mathematique, nouvelle serie, tome 60 (74), 1-4, 1996

[8] Z. Ognjanović, M. Rašković. Some Probability Logics with New Types of Probability Operators. J. Logic Computat. Vol. 9, No. 2, 181-195, 1999

[9] Z. Ognjanović, M. Rašković. Some First-Order Probability Logics. Theoretical Computer Science 247 (1-2), 191-212, 2000

[10] Z. Ognjanović, T. Timotijević, A. Stanojević. Database of Papers about Probability Logics. Mathematical Institute Belgrade. http//problog.mi.sanu.ac.yu/. 2005

[11] Z. Ognjanović, Z. Marković, M. Rašković. Completeness Theorem for a Logic with Imprecise and Conditional Probabilities. Publications de l'institute mathematique, nouvelle serie, tome 78 (92), 35-49, 2005

[12] Z. Ognjanović, A. Perović, M. Rašković. Logics with the Qualitative Probability Operator. Logic Journal of the IGPL. doi:10.1093/jigpal/jzm031, 2007

[13] M. Rašković. Classical Logic with some Probability Operators. Publications de l'institute mathematique, nouvelle serie, tome 53 (67), 1-3, 1993

[14] M. Rašković, Z. Ognjanović. A First Order Probability Logic $LP_Q$. Publications de l'institute mathematique, nouvelle serie, tome 65 (79), 1-7, 1999

[15] M. Rašković, Z. Ognjanović, Z. Marković. A Logic with Conditional Probabilities. In J. Leite and J. Alferes (Eds.), 9th European Conference Jelia'04, lecture notes in artificial intelligence (LNCS/LNAI), 3229, 226-238, Springer 2004

[16] D. Scott. Measurement Models and Linear Inequalities. Journal of Mathematical Psychology, 1, 233-247, 1964

[17] W. van der Hoek. Some Considerations on the Logic $P_F D$: a Logic Combining Modality and Probability. Journal of Applied Non-Classical Logics, 7 (3), 287-307, 1997

[18] M. P. Wellman. Some Varaities of Qualitative Probability. Proc. of 5th International Conference on Information Processing and the Management of Uncertainty, Paris 1994