Diagrams based on the Hexagonal and Triangular Grids

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Abstract: In this study we show that sometimes it is worth to consider a non-traditional regular grid in diagrams and proofs by diagrams instead of the traditional rectangular arrangement. We refer to the hexagonal and triangular grids as non-traditional regular grids. Particularly, we show that not only triangular numbers can be arranged in triangular shape, but also, it is natural to represent the square numbers via triangles on the triangular grid. On the other hand, we show that hexagonal arrangement is useful to represent binomial coefficients, and also to solve jug puzzles. The triangular and hexagonal grids are dual of each other and share many properties including symmetric properties. Based on our study, we propose to consider them more frequently in education and diagrammatic applications as well. This would be fruitful also for critical thinking, design and problem solving in mathematics, engineering and informatics including artificial and computational intelligence.

Keywords: Computer-aided Education; Non-traditional Diagrams; Geometric Proof by Diagrams; Human-Computer Interaction; Data Visualisation; Number of Shortest Paths; Figural numbers

1 Introduction

In mathematics and many disciplines that uses (applied) mathematics and/or computations, x-y plots are essential to display results. Plots and diagrams play significant importance in education and they are main tools of human computer interactions as well. Based on the well-known Cartesian coordinates, they are used to display, examine and analyze functions. They have both theoretical and practical importance. More generally viewing, diagrams are useful tools in various disciplines. On the one hand, they are used in geometric and other mathematical/ computational proofs and problem solving including artificial and computational intelligence, as well as in engineering, mechanics and other physics related disciplines [3]. On the other hand, they are used to represent data in many other fields including statistics, psychology, sociology and in industry, just to mention a few. A diagram could summarize plenty of observations and statements including comparisons, developments, etc. [19]. There are two main classes of diagrams, continuous and discrete diagrams, depending on the nature of the data (and the domain from where the data are taken) that is displayed by them. In this paper, we consider discrete diagrams, in the sense that we are using discrete sets (either finite sets or maybe a portion of some discrete sets, e.g., the set of integers) as domain and also the values are from a discrete set, e.g., nonnegative integers.

There is a wide variety of diagrams that are frequently used for various aims. One of the most known diagrams are the column/bar diagrams which are appropriate to represent some data on a (finite subset of a) discrete domain. In these diagrams, depending on the "dimension" of the domain, the columns are usually arranged in a line, Fig. 1 (left) or, in case of two-dimensional domain, in a rectangular format, Fig. 1 (right). In this paper, we show that sometimes other arrangements could be more beneficial (even if, the most popular and known software tools do not support to create such diagrams in an easy way).

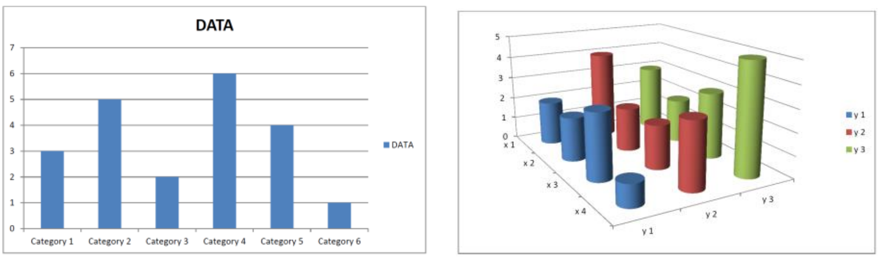

Figure 1
Usual format of column diagrams having one-dimensional (maybe linear) domain (left) and two-dimensional domain (right)

The structure and the content of the paper are as follows. In the next section we recall some details of the hexagonal and triangular grids with their symmetric coordinate systems. The triangular grid consists of equilateral triangles and forms a closed surface without gaps by the rotation of every second triangle. The hexagonal grid consists of equilateral hexagons, which also form a closed surface. Special coordinate systems, which are recalled, with coordinate triples are used for both grids to represent the movement along the main axes. Due to restrictions that the sum of the coordinates in the triplet is a fixed value, these are basically 2-dimensional. In Section 3 we show figural numbers and especially, as a novel result, we show how the triangular grid can represent squares. Section 4 is about the binomial coefficients and their beneficial representation on the hexagonal grid implied by the detailed comparison of the square and hexagonal grid approaches. In Section 5 jug problems are represented on a non-traditional grid. These represent various usages of diagrams to aid thinking about mathematical and computational problems. Finally, in Section 6 we conclude the paper by motivating teachers and researchers to use non-rectangular grids for the representation of suitable problems. Also, we sketch the use of non-cubic grids in the 3-dimensional domain.

On the one hand we give some really new material, e.g., about the square numbers in the triangular grid, but we also recall some known things to highlight them for this new perspective we propose here, e.g., jug problems. The binomial coefficients are usually represented by the Pascal's triangle (either on the square or on the hexagonal grid), we give a comparison by highlighting the "hidden" difference and we give an argument supporting the hexagonal grid approach. Up to the knowledge of the author there is no such work exists in the literature where the regular grids are compared by this perspective, i.e., how can we use them in education or in human-computer interaction to solve mathematical puzzles and problems and/or to use them to do and display inductive geometrical proofs in number theory. On the other hand, there are various algorithms known for these regular grids including some in image processing and computer graphics [6, 31], while the non-traditional grids are also frequently used in computer games [28]. The main aim of the paper is to open the eyes of the researchers, teachers and problem solvers by showing that state space representations, computer programs as well as mathematical descriptions and proofs can be done in a simpler way by finding a more suitable non-traditional grid based representation of the problems.

2 Non-Traditional Regular Grids: the Hexagonal and the Triangular Grids

In this section we recall and present the three regular two-dimensional grids. Each of them is based on a regular polygon, a tile (that is of fixed size, but maybe rotated), that is able to tessellate the entire plane without overlaps and holes in a monohedral and isogonal way (that is, the tiles are identical, and also the crossing points of the grid are identical) in edge-to-edge manner (that is if two tiles share more than one point on their boundaries, then they share a full edge, i.e., the sides of the respective polygons). There are three regular polygons that can be used for this aim; they are the squares, the equilateral triangles and the regular hexagons. We name the grids by the tiles that are used, and thus, we discuss the square grid, the triangular grid and the hexagonal grid. Sample parts of these grids are shown in Fig. 2. It is an important fact that grids have duals, which are obtained by exchanging the roles of regions and vertices of their graph, i.e., by putting points to the midpoints of the regions/tiles and connecting those by lines, which belong to tiles that have an edge in common. In this sense, the dual of the square grid is a square grid, but the hexagonal and triangular grids are dual grids of each other.

Whenever one uses a grid, it is essential to have an easy-to-use elegant scheme to address the parts used, here the tiles. For the square grid the well-known Cartesian coordinate system fits, by restricting the coordinates to integers (Fig. 2, left). Since the Cartesian system already appears in elementary school, and the pupils use square-patterned exercise books, the square grid is widespread. Whenever somebody is talking about the "digital world" including digital image processing, computer vision, printing, etc., he or she almost always thinks only of the square grid. However, as we also argue here, the world is not arranged in this way, one may consider other grids that could have some more beneficial properties depending on the application.

The hexagonal and triangular grids have also easy-to-use symmetric coordinate systems. The tiles of the hexagonal grid can be addressed by zero-sum integer triplets based on Her's papers [11, 12]. See Fig. 2 (middle). Lanes are those sequences of tiles that share a fixed coordinate value. They play similar role in this grid, as columns and rows play in the square grid. As the coordinate system reflects well the symmetry and the structure of the grid, the coordinate difference of two neighbor tiles is +1 and –1 in two of the coordinates, while the third coordinate value is shared for these tiles.

The analogous symmetric coordinate frame for the triangular grid is shown in Fig. 2 (right) [20-22, 30]. Lanes are also defined analogously to the hexagonal case. Notice that, the triangular grid is, in fact, not a point lattice, there are two different orientations of the tiles, and vectors connecting differently oriented tiles do not translate the grid into itself. The coordinate system, however, is reflecting the two types of tiles by using 0-sum triplets for even tiles (△ oriented in our figures) and 1-sum triplets for odd tiles (▽, respectively). The terms even and odd tiles fit well in this context, as all neighboring tiles (that share a side) is of opposite type than the original tile, and their coordinate triplets are obtained by adding (subtracting) 1 for a coordinate value, if the original tile is even (odd, respectively). The triangular and hexagonal grids are closely related and share some symmetric properties.

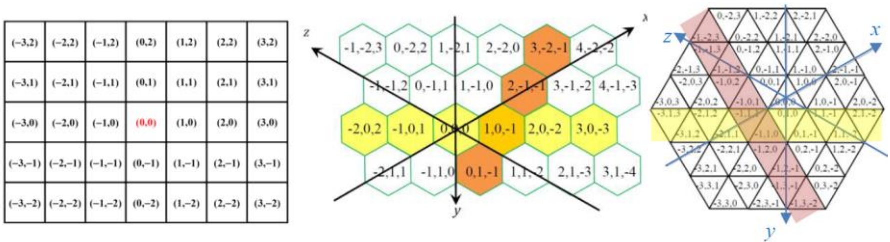

Figure 2

Some parts of the three regular tessellations: the square grid (on the left), the hexagonal grid (in the middle) and the triangular grid (on the right) with their symmetric coordinate systems. Lanes are also highlighted in the latter two grids.

We underline again that since the hexagonal and triangular grids are graph theoretic duals of each other, there is a confusion of their names also in the scientific literature. In applications similar to communication networks, usually the grid points, the crossing points of the edges are addressed and used, while in graphical algorithms, in image processing and analysis, the hexagons/triangles are used and addressed as pixels. Here, in the paper, we use the convention that the grid is named after its tiles in the tessellation, which fits the latter approach. On the other hand, sometimes the name honeycomb grid is also used in the literature, e.g., in relation to networks [30], and it usually refers to the triangular grid in our terminology.

3 Triangular Numbers and Squares

3.1 Usual Diagrammatic Representations (based on Hexagonal and Square Grids)

Already the ancient Greeks have defined and used figural (also called figurate or figurative) numbers which were defined as the number of small stones arranged in a geometric form. The most known such numbers are the squares, but also the triangular numbers have a large literature [7]. Fig. 3 shows some of these numbers in the usual diagrammatic representations.

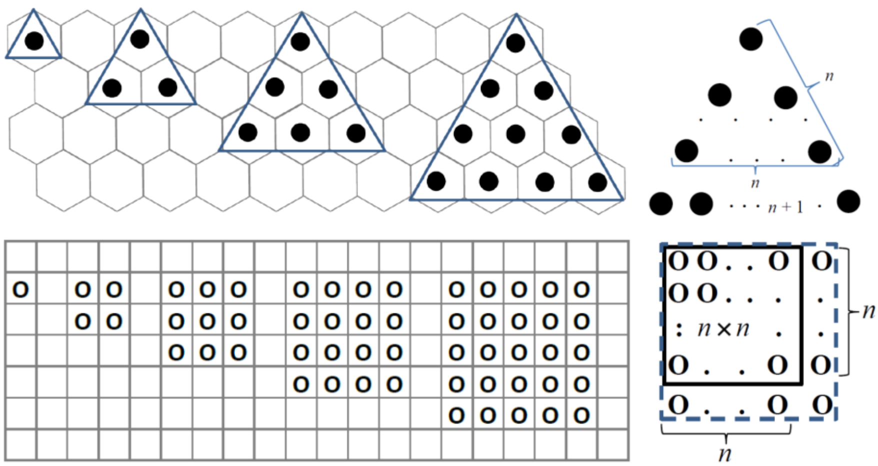

Figure 3

Some triangular numbers (top) and squares (bottom) as figurate numbers

While the square numbers $n^{\square}$ can be computed as squares of natural numbers, i.e., there is a well-known operation that yields exactly to those numbers $n^{\square} = n^2$, there is no direct mathematical operation is used to get the triangular numbers. On the other hand, by the structure of the shape, it is easy to see that

$$n^{\triangle} = \sum_{i=1}^{n} i \tag{1}$$

which in fact results

$$n^{\triangle} = \frac{n(n+1)}{2} \tag{2}$$

for a triangle having *n* stones in each of its sides.

As one can observe, the stones are put in a triangle shape, but their neighborhood refers to the hexagonal grid (as we showed by the grey color). On the right side of the figure it is also shown how the next number can be obtained in the general case, e.g., the proof of

$$(n + 1)^2 = n^2 + 2n + 1 \tag{3}$$

is displayed. In the next subsection, we investigate the number of stones in a triangle shape when the underlying grid is the triangular one.

3.2 Representing Squares on/with Triangles

In the previous part we have shown that triangular numbers and square numbers are usually represented by hexagonal and rectangular arrangements, respectively. One may ask, what about if the triangular grid is used as a basis of the representation. In this subsection we investigate this topic and show its significance. We also give geometric proofs for some interesting number theoretic facts by applying our diagrams.

Figure 4 shows some of these diagrammatic shapes. On the right hand side of the figure we also show the "construction" for the proof of the next statement.

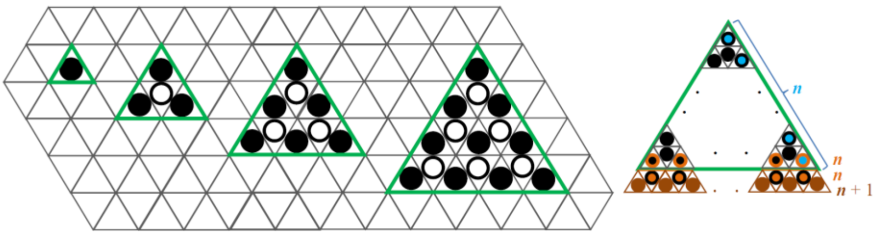

Figure 4
Square numbers in triangular shapes in the triangular grid. Even pixels are with black stones, odd pixels are with white stones (left). The number of light blue stones/balls (on even pixels) is n on the right, similarly there are n orange bordered balls (on even pixels) and the same number of black bordered orange balls (on odd pixels). The number of brown balls is n + 1.connections

Proposition 1. *The number of stones in the triangle having side-length n on the triangular grid is exactly n^2.*

Proof. It goes by induction.

Base case: n = 1,

$$1^2 = 1, \tag{4}$$

and the first (leftmost) figure contains exactly 1 stone.

Hypothesis: the equilateral triangle with side length k contains exactly

$$k^2 \tag{5}$$

stones (triangles).

Proof of inheritance: we obtain the equilateral triangle with side length $k + 1$ by adding some triangles of the next lane, especially, number k odd and number $k + 1$ even triangles, altogether $2k + 1$ triangles are added. This with the hypothesis (5) gives

$$k^2 + 2k + 1 \tag{6}$$

triangles which number is the same as

$$(k + 1)^2. \tag{7}$$

□

Moreover, from Fig. 4, by observing that in fact, the black and white stones represent two consecutive triangular numbers, i.e., $n^{\triangle}$ of even triangles (with black stones on them) and $(n-1)^{\triangle}$ odd triangles with white stones on them ($n > 1$), we can deduce the following result by having this new diagrammatic proof.

Proposition 2. *The sum of two consecutive triangular numbers is always a square, i.e.,*

$$(n+1)^{\triangle} + n^{\triangle} = (n + 1)^2. \tag{8}$$

If one thinks about subsets of the grids having these special shapes, they can also be defined by their coordinates in an elegant way, and thus, the triangular grid should not be discriminated against the square grid. While a square with "area" of n^2 can be defined on the square grid with the pixels $0 \le x,y \le n$, a triangle with of n^2 triangle pixels is defined, e.g., by $x,z \le 0$ and $y \le n$. Further, $|x|,|y| \le n$, on the square grid defines square with side-length $2n+1$ and, thus with $(2n+1)^2$ pixels. On the triangular grid $|x|,|y|,|z| \le n$ defines triangle with side-length $3n+1$ and, thus with $(3n+1)^2$ pixels. In some cases, and very often in computerized scenario, one can limit the values of the variables according to the fact that only a finite segment of the discrete space is really needed. The description we gave above is also helpful from this point of view, showing the relation of the number of possible states and the possible values of the variables. As we have shown, the triangular grid gives a nice analogy of the square grid even in diagrammatic thinking and proof techniques based on the symmetric coordinate system recalled in this paper.

As we can see, representing the squares, as well as the proof that the squares are represented by the equilateral triangles on the triangular grid is as natural as their representation and proof on the square grid. Actually, the result we have presented in this section is related to the coarsening property of the square and triangular grids [9] that can be used in various applications, e.g., imaging and cartography. This property says that a pixel can be divided into equal-sized same-type regular, but smaller pixels. This refinement in both of these grids means that a pixel can be cut to n^2 pixels (for any natural number n). For the sake of completeness we should note that the hexagonal grid does not have this property, a hexagon tile cannot be cut to smaller identical regular hexagons.

In the next sections we present such examples when the usage of non-traditional grids has a real advantage.

4 Diagrams of Binomial Coefficients

It is a well-known way to compute the binomial coefficients by the Pascal's triangle. Let us go a little bit closer and see what should be the underlined grid for, e.g., a column type diagram showing some finite segment of the Pascal's triangle. As usual, the numbers of the triangle are written in lines below each other. The next row is written below the previous one and the values are inserted in the space between the values of the of the previous row, one may think about two options: either to put the numbers in a square grid, but rotate the grid by 45°; or to arrange the numbers according to the hexagons of the hexagonal grid (see Fig. 5 for both options). Here, we are arguing for the latter option. For this, let us see how the "neighbor" numbers are related to each other (see the arrows on the figures). To work with binomial coefficients, let us have n marbles in a row such that k of them are (identical) red and the rest $n - k$ are (identical) blue. The binomial coefficient $\binom{n}{k}$ gives the number of ways to put our marbles in a sequence. The neighboring elements represented by the blue solid arrow show scenarios when we got a new blue marble, the red solid arrow shows the case when a new red marble is provided. Broken arrows show that we have removed a blue or a red marble, respectively. Yellow arrows show that how it could be that we got one extra marble and now we have, e.g., $n + 1$ marbles such that k of them is red ($n = 5$ and $k = 4$, in the left side of the figure). In this way, the two yellow arrows should be considered together. Consequently, the addition rule can be applied showing the identity

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \tag{9}$$

for $n \geq k > 0$. On the left, on the square grid we have considered all four neighbors of a cell. However, the neighborhood shown by green color on the right also carries some information. The green solid (broken) arrow shows the change of the scenario by changing, i.e., recoloring one of the blue marbles to red (vice versa, respectively). Let us consider the part of the grid with these numbers such that the top hexagon is addressed by $(x,y,z) = (0,0,0)$ meaning that we have x red, z blue and altogether y marbles. We are using actually that sextant of the grid where $y \geq 0$ and $x,z \leq 0$. Thus, to have a valid meaning of the coordinates in our marble scenarios, we use the absolute values of the coordinates presented in Fig. 2 (middle). In this way, it is straightforward to see that the blue solid arrow direction is increasing the y and z coordinates (adding a blue marble), the red solid arrow direction is increasing the x and y coordinates and the green solid arrow direction is increasing the x and decreases the z. The unmentioned third coordinate does not change in these movements. Broken arrows act in opposite way, respectively. Moreover, the directions of the green arrows, the horizontal lanes of the hexagonal grid gives all the possibilities when we have exactly *n* marbles, the values in these lanes sum up to 2^n showing that if any marble could be red or blue, we have actually, that many possibilities to list the *n* marbles (two color, two possibilities for each marble and multiplication rule is applied). Actually there are various places in the literature, e.g., [13], where the binomials are written not with two, but with three values, such that the sum of the two values in the lower part is the upper value. This writing, i.e., $\binom{n}{k, n-k}$ reflects well that behind the scene the hexagonal structure should be understood. Moreover, this notation also underlines the other important identity among binomial coefficients, i.e., the symmetry

$$\binom{n}{k, n-k} = \binom{n}{n-k, k}. \tag{20}$$

The lanes of the grid along two of the three lane directions reflect the scenarios when the numbers of the red and blue marbles are fixed to *k* and *n* – *k*, respectively, and finally in the green arrows direction, the sum, the total number of marbles is fixed (to *n*). Thus, it is beneficial to design a column diagram for the binomial coefficients with a two-dimensional domain by arranging the columns in a hexagonal structure.

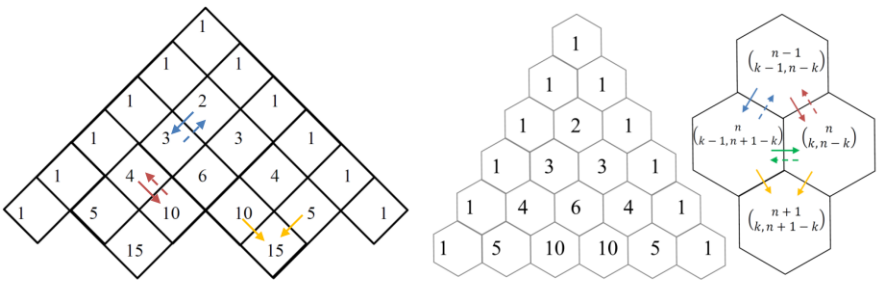

Figure 5

Binomial coefficients represented on grids with their relations

Note that both in the square and in the hexagonal grid cases, the binomial coefficients on the pixels show the number of shortest paths to the given pixel from the top most pixel, where a path is going through on neighboring pixels.

The application of the hexagonal grid for binomial coefficients served as an argument in favor of the hexagonal approach.

5 Solving the Jug Problem

In problem solving in general, as well as, particularly, in Artificial Intelligence, it is crucial how the problem is represented (in mathematical, formal way and also on the computer) [15, 16]. By having an elegant representation, the problem and also its solution can be simplified a lot. In this section, we present such an example, when a non-traditional grid could help a lot.

As we have seen the hexagonal grid is opt to represent triplets with a fixed sum. Although in Fig. 2 (middle) the coordinate system uses 0-sum triplets, as it is shown, e.g., in [22], any fixed sum integer triplets can be used. In the most known jug problems three jugs with various capacities are used and the sum of the water in them is fixed by the initial condition, e.g., the largest jug is full initially, the others are empty. The task is to measure a given amount of water that is usually not identical to the capacity of any of the jugs used. The rule that allows one to pour water from a jug (let it be called source jug) into another (target jug, respectively) is as follows: let the amount of water in the source jug be s, and let the capacity of the target jug be c, while the actual amount in it is t ($c \geq t$); then the player can pour (transform) exactly

$$\min\{s, c - t\} \tag{11}$$

amount of water. The amount either fits the target jug or this amount empties the source jug. The problem is well known in Artificial Intelligence and also plays an important role in the teaching of algorithmic solutions for artificial intelligence problems. Here, we present the specific problem with jugs of capacity 5, 3 and 2 liters and with initial state having 5 liters of water in the largest jug. The goal could be to obtain either 1 or 4 liters of water in a jug (to obtain 2 or 3 liter is obvious by the size of the two other jugs). Now instead of the hexagons, we use the dual representation and use the gridpoints to represent the possible states of the problem.

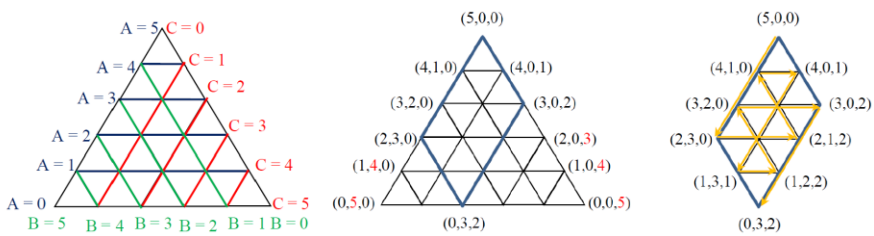

Figure 6

Diagrammatic representation and solution of the jug problem with jugs with 5, 3 and 2 liters, starting with the largest jug full

Fig. 6 on the left shows the grid, where triplets (A,B,C) represent the amount of waters in the jugs of size (5,3,2), respectively. Lanes with different color show the states where the amount in one of the jugs is fixed. Some of the states of the left figure are excluded because of the capacities of the jugs. The blue parallelogram in the middle represents the given constraints (red values show the coordinates representing invalid values). One can see that only triplets of the lanes $B \in \{0,1,2,3\}$ and $C \in \{0,1,2\}$ are allowed. By the constraint the problem have 12 possible states.

An operation from a state to a state is to move in the graph along a lane until one reaches the border of the parallelogram. The yellow arrows represent a path which touches all the 10 states that can be reached in the problem with the specified operations:

(5,0,0) → (2,3,0) → (2,1,2) → (4,1,0) → (4,0,1) → (1,3,1) → (1,2,2) → (3,2,0) → (3,0,2).

The direction (and orientation) of the arrows, the operations applied show which two jugs were considered (the third one has a fixed value along the direction considered, i.e., the lane embedding the arrow), the direction of the arrow shows from where the water is poured to where:

- Horizontal arrows from left to right show operation to pour water from B (3 liter capacity jug) to C (2 liter capacity jug), while opposite direction arrows show operation to move water from C to B; in these movements jug A, the 5 liter capacity jug does not change its water contents (these operations are moves on the blue lines on Fig. 6, left).

- Moving along the green lanes of Fig. 6 (left), down-left direction arrows represent operation from A to C, right-up direction represents from C to A, left-up direction represents from B to A, and finally, right-down direction arrows (red lane directions) stand for operations that move water from A to B.

Since all the reachable states are touched in the previously described path, the solution of any problem can be read, e.g., 1 liter can be obtained after 2 operation applications, while 4 liter can be obtained in the largest jug after 3 operation applications. One can also see that some states cannot be reached, i.e., there is no way to get (3,1,1) and (2,2,1) if none of them were the initial state. We note that in our special case, the operations of our path are invertible, one can go back to some earlier stage, which is however, not the case in many analogous jug problems. Related problems are nicely studied in [5] with some geometric thoughts, using reflections, i.e., when an operation is not represented by the side of the parallelogram, its arrow touches the boundary in a state, and if this state is not the corner of the parallelogram, then the “reflection”, similarly as the light is reflected on a mirror gives the next operator that can be used. In this way, the chain of operations used can also be seen as a billiard ball experiment by listing the states of the border touches of the ball when it reflects ideally on the sides of the parallelogram (when the initial state is not in a corner of the parallelogram).

A solution of the jug problem with the dual representation of the hexagonal grid is depicted clearly showing that the usage of non-rectangular grids for the representation of problems can have advantages compared to rectangular grids.

In general, we can state that a computer program can be much shorter and faster, moreover it is easier to design/follow it if the appropriate representation is used. However, in the program it may not necessarily be stored and will use a third variable if three variables have a fixed-sum: For instance, in the case of the hexagonal approach above, one may store only the actual values of the contents of the first two jugs, A and B as, let us say, x and y, and from those the content of the third jug, C, can always be determined as $5 - x - y$. The important thing is not actually, the number of variables used in the implementation, but the picture, the diagram of the problem behind the scene in which here we used the third value even if it is dependent on the first two values. With these comments we have arrived to the last part of the paper.

Discussion, Conclusions and Further Thoughts

In this paper, it was shown that it is worth to consider a non-traditional representation of a problem. Diagrams based on non-traditional two-dimensional regular tilings were discussed. We have started with the comparison of the representations of triangular numbers in the hexagonal grid with square numbers in the rectangular grid as a motivational example. We have shown that a different view could be beneficial, it may give a proof or a technique with the same simplicity or even simpler way than using the rectangular representation of the problem. Especially, we have shown that equilateral triangles on the triangular grid represent the square numbers, and the difference of two consecutive square numbers can easily be seen on their diagrammatic representations. We have shown that a (general) binomial coefficient has six neighbors, each with a specific meaning, thus the hexagonal grid is more comfortable to use for representing Pascal's triangle than the square grid. Finally, the jug problem is recalled, and again, even though the states of these problems have only two parameters, it is fruitful to use a representation that based on three variables but have a fixed sum. Our results are not only important in educational purpose, e.g., in computer aided teaching, but also play significant role for system designers and researchers, when thinking about the best ways of human computer interaction or data visualisation in their fields. As we have clearly shown, using problem-specific grids makes sense and simplifies many issues, in particular some representations. One of the most important open questions here is to characterize when it is better to use, e.g., two independent variables and not naming and referring the others or to use three named variables with a constant sum (see the jug problem). Mathematically these descriptions are equivalent, but from simplicity, from diagrammatic and reasoning point of view they are different. There is no general rule found yet, but one needs to think carefully when choosing the representation of the problem. The reasoning, the problem solving, the algorithm and the computer program could be easier and simpler, if additional meaningful values are computed and used, even if they are determined by the free variables.

In the paper we have treated only discrete problems with discrete diagrammatic tools, however, as the Cartesian coordinates can be used for the whole plane, also the hexagonal and triangular coordinates have such extensions [1, 12, 23], thus in the future continuous diagrams can also be expanded by changing their domain from the rectangular to the hexagonal or to the triangular grids. On the other hand, there are many other useful two-dimensional grids, e.g., some semi-regular grids and their duals (cf. [2, 4, 10, 17, 18, 27, 29]).

The hexagonal grid is optimal in the sense that the smallest length of the grid lines is used to create a grid, this is why the bees use this structure and this is where the name honeycomb comes for this type of drawings. Also, among the two-dimensional grids the hexagonal grid provides the best packing density (by putting equal-sized disks, the most efficient way is to arrange them according to a hexagonal grid that gives the places of the disks as the inner circles of the hexagons). In three dimensions, it is not the cubic that grid gives the best sphere packing structure, but, e.g. the face-centered cubic grid [14] which can be seen, in this term, as the three-dimensional generalization of the hexagonal grid. On the other hand, in the case of the hexagonal grid it holds that, for each pixel, if they have a common boundary point, they share a full side, thus, there is only one type of neighbor relation among the pixels. This is not true for the square and triangular grids which may lead to some topological paradoxes. For instance, considering also diagonal neighbors on the square grid, one may have two lines which cross each other without a common tile: the two diagonals of a chessboard are built up by black and white squares, respectively, thus clearly they have no common tile (square). Considering only the closest 4 neighbors of each square, one may “draw” a closed curve which may have multiple insider regions causing another topological paradox, i.e., the Jordan curve theorem does not hold for this digital scenario. There is no such problem on the hexagonal grid, the digital (grid) analogue of the Jordan curve theorem holds. The cubic grid has also various pairs of cubes that share point(s) on their boundary, but they do not share a full square side, thus it is also paradoxical, and thus, there are various difficulties when one wants to use it in three-dimensional graphics and imaging. The body-centered cubic grid, in three dimensions, gives such a tessellation of the space that any voxel (Voronoi cell of the tessellation) has only neighbors where either a full hexagon side or a full square side is shared. In this way objects made in this grid will stay connected and do not fall apart. In this sense the body-centered cubic grid can be used as a generalization of the hexagonal grid to three dimensions [14]. Finally, the three-dimensional analogue of the triangular grid is the diamond grid representing the structure of the connections of the Carbon atoms in the diamond crystal [8, 24]. Some efficiently used coordinate systems for these three-dimensional grids are provided in [25, 26], some of them use more than 3 coordinates, but restrictions on the sum of the values.

We believe that diagrammatic thinking and applications of diagrams based on the non-traditional grids is fruitful, but less known areas for the community. As we have shown by our case studies, there are various problems where the use of non-traditional grids gives some advantages. With this paper, we invite everybody who is interested to contribute to this direction as well.

References

[1] Abuhmaidan, K., Aldwairi, M., Nagy, B.: Vector Arithmetic in the Triangular Grid, Entropy 23(3) paper 373 (2021)

[2] Borgefors, G.: A semiregular image grid. J. Vis. Commun. Image Represent. 1(2): 127-136 (1990)

[3] Chang M. D., Wetzel J. W., Forbus K. D.: Spatial Reasoning in Comparative Analyses of Physics Diagrams. In: Freksa C., Nebel B., Hegarty M., Barkowsky T. (eds.) Spatial Cognition IX. Spatial Cognition 2014. LNCS, Vol. 8684, pp. 268-282, Springer, Cham (2014)

[4] Conway, J. H., Burgiel, H., Goodman-Strass, C.: The symmetries of things. AK Peters (2008)

[5] Coxeter, H. S. M., Greitzer, S. L.: Geometry revisited. The Mathematical Association of America, Washington D.C. (1967)

[6] Deutsch, E. S.: Thinning algorithms on rectangular, hexagonal and triangular arrays. Comm. ACM 15, 827-837 (1972)

[7] Deza, E., Deza, M. M.: Figurate numbers. World Scientific, Hackensack, NJ, (2012)

[8] Eppstein, D.: Isometric Diamond Subgraphs. In: Tollis, I. G., Patrignani, M. (eds.) GD 2008: Graph Drawing, 16th International Symposium, LNCS, Vol. 5417, pp. 384-389, Springer, Heidelberg (2009)

[9] Gaspar, F. J., Gracia, J. L., Lisbona, F. J., Rodrigo, C.: On geometric multigrid methods for triangular grids using three-coarsening strategy. Applied Numerical Mathematics 59(7), 1693-1708 (2009)

[10] Goodman-Strauss, C, Sloane, N. J. A.: A coloring-book approach to finding coordination sequences. Acta Crystallographica Section A: Foundations and Advances 75(1), 121-134 (2019)

[11] Her, I.: A symmetrical coordinate frame on the hexagonal grid for computer graphics and vision. ASME. J. Mech. Des. 115(3), 447-449 (1993)

[12] Her, I.: Geometric transformations on the hexagonal grid. IEEE Transactions on Image Processing 4(9), 1213-1222 (1995)

[13] Hilton, P., Pedersen, J.: Relating Geometry and Algebra in the Pascal Triangle, Hexagon, Tetrahedron, and Cuboctahedron Part I: Binomial Coefficients, Extended Binomial Coefficients and Preparation for Further Work. The College Mathematics Journal 30(3), 170-186 (1999)

[14] Ibáñez, L., Hamitouche, C., Roux, C.: Ray-tracing and 3D Objects Representation in the BCC and FCC Grids. In: Ahronovitz, E., Fiorio, C. (eds.) DGCI'97: 7th International Conference on Discrete Geometry for Computer Imagery, LNCS, vol. 1347, pp. 235-242, Springer, Heidelberg (1997)

[15] Kósa, M., Nagy, B., Pánovics, J.: Performance Analysis of Search Algorithm Depending on the State Space Representation, 6th MaCS, 6th Joint Conference on Mathematics and Computer Science, July 2006, Pécs, Hungary

[16] Kósa, M., Nagy, B., Pánovics, J.: Megoldáskereső algoritmusok hatékonyságának vizsgálata az állapottér-reprezentáció függvényében (Performance Analysis of Search Algorithm Depending on the State Space Representation) (2006), SzámOkt 2006 Számítástechnika az oktatásban XVI. nemzetközi konferencia, 16th International Conference in Computer Science and Education, Sovata, Romania, 76-81

[17] Kovács, G., Nagy, B., Turgay, N. D.: Distance on the Cairo pattern. Pattern Recognition Letters 145, 141-146 (2021)

[18] Kovács, G, Nagy, B., Vizvári, B.: Weighted Distances and Digital Disks on the Khalimsky Grid – Disks with Holes and Islands. J. Math. Imaging Vis. 59(1), 2-22 (2017)

[19] Loockwood, A.: Diagrams: A Visual Survey of Graphs, Maps, Charts and Diagrams for the Graphic Designer. Watson-Guptill, New York (1969)

[20] Nagy, B.: Finding shortest path with neighbourhood sequences in triangular grids. In: International Symposium on Image and Signal Processing and Analysis conjunction with 23rd International Conference on Information Technology Interfaces 2001, pp. 55-60, IEEE, Pula (2001)

[21] Nagy, B.: A symmetric coordinate frame for hexagonal networks. In: Proc. of IS-TCS'04: Theoretical Computer Science – Information Society, Ljubljana, Slovenia, pp. 193-196 (2004)

[22] Nagy, B.: Generalized triangular grids in digital geometry. Acta Mathematica Academiae Paedagogicae Nyíregyháziensis 20, 63-78 (2004)

[23] Nagy, B., Abuhmaidan, K.: A continuous coordinate system for the plane by triangular symmetry. Symmetry 11(2), paper 191 (2019)

[24] Nagy, B., Strand, R.: Neighborhood Sequences in the Diamond Grid. In: Barneva, R. P., Brimkov, V. E. (eds.) Image Analysis – From Theory to Applications. Proceedings of IWCIA 2008: 12th International Workshop on Combinatorial Image Analysis, Special Track on Applications, pp. 187-195, Research Publishing (2008)

[25] Nagy, B., Strand, R.: A connection between $\mathbf{Z}^n$ and generalized triangular grids, ISVC 2008, 4th International Symposium on Visual Computing, Lecture Notes in Computer Science - LNCS 5359 (2008), 1157-1166

[26] Nagy, B., Strand, R.: Non-Traditional Grids Embedded in Z^n. Int. J. Shape Model. 14(2), 209-228 (2008)

[27] Radványi, A. G.: On the rectangular grid representation of general CNN networks. International Journal of Circuit Theory and Applications 30(2-3), 181-193 (2002)

[28] Red Blob Games, https://www.redblobgames.com/ [visited 20 June, 2021]

[29] Saadat, M. R., Nagy, B.: Digital Geometry on the Dual of Some Semi-regular Tessellations. In: Lindblad J., Malmberg, F., Sladoje, N. (eds.) DGMM 2021: Discrete Geometry and Mathematical Morphology, 1st IAPR International Conference, LNCS 12708, 283-295 (2021)

[30] Stojmenović I. Honeycomb Networks: Topological Properties and Communication Algorithms. IEEE Trans. Parallel Distributed Syst. 8(10), 1036-1042 (1997)

[31] Wüthrich, C. A., Stucki, P.: An algorithmic comparison between square- and hexagonal-based grids. CVGIP Graph. Model. Image Process. 53(4), 324-339 (1991)