Sturm–Liouville problem and I–Bessel sampling

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Abstract: *The main aim of this article is to establish summation formulae in form of the sampling expansion series building the kernel function by the samples of the modified Bessel function of the first kind I_ν, and to obtain a sharp truncation error upper bound occurring in the derived sampling series approximation. Summation formulae for functions $I_{\nu+1}/I_\nu, 1/I_\nu, I_\nu^2$ and the generalized hypergeometric function ${}_2F_3$ are derived as a by–product of these results.*

The main derivation tools are the Sturm–Liouville boundary value problem and various properties of Bessel and modified Bessel functions.

Keywords: *Bessel function of the first kind J_ν, modified Bessel function of the first kind I_ν, sampling series expansions, Sturm–Liouville boundary value problems, generalized hypergeometric function ${}_2F_3$, Fox-Wright generalized hypergeometric function ${}_p\Psi_q^*$, sampling series truncation error upper bound.*

1 Introduction and motivation

The historical background of sampling theorems, various applications in many branches of science and engineering, especially in signal analysis and reconstruction and/or its up-to-date results in different areas of mathematics like approximation theory and interpolation are well–covered among others by Jerri's "IEEE 1977 paper" [13], by survey articles of Khurgin–Yakovlev [14] and Unser [24], by the monographs of Higgins [9], an edited monograph by Higgins and Stens [10], the book by Seip [22] and numerous references therein. Thus, by skipping an outline of the facts from the aforementioned references we can focus on our main goal – establishing the I–Bessel sampling expansion result *via* the appropriate Strum–Liouville boundary value problem and the related sampling expansion series truncation upper bound, which yields the precise convergence rate in this kind of approximation procedures.

Here and in what follows *$\mathscr{B}$–Bessel sampling* is called a sampling expansion procedure for some input function f, when the underlying sampling kernel function is built up in terms of samples of $\mathscr{B}$ being a Bessel or modified Bessel function, and the sampling nodes correspond to the zeros b_k of $\mathscr{B}$ used in the expansion formula.

For instance, Kramer considered J–Bessel sampling as an illustrative example for his theorem [17] which generalized the Whittaker–Shannon–Kotel'nikov (WKS) sampling theorem [30]. More precisely, Kramer derived the following summation formula:

$$f(t) = 2J_m(t) \sum_{k\in\mathbb{Z}} \frac{j_{m,k}\, f(j_{m,k})}{(j_{m,k}^2 - t^2)\, J_{m+1}(j_{m,k})}, \qquad J_m(j_{m,k}) = 0\,.$$

Before Kramer, we have to mention Weiss [29] who arrived at the same result for $k = 2$, and also Whittaker who first discussed a very similar sampling expansion [30]; see also [31, p. 439, Eq. (17)]:

$$f(t) = \frac{2\sqrt{t}}{\pi}\, J_\nu(\pi t) \sum_{k\geq 1} \frac{\sqrt{t_k}\, f(t_k)}{J_{\nu+1}(\pi t_k)(t_k^2 - t^2)}, 0 < t < \infty,$$

where $\{\pi t_k\}$ are positive zeros of $J_\nu(\pi t)$, $\nu \geq \frac{1}{2}$. It is worth mentioning that a recent article by Jankov Maširević *et al.* [12] is devoted mainly to Y–Bessel sampling, where Y stands for the Bessel function of the second kind.

On the other hand, the sampling theorem is related to Sturm–Liouville boundary value problems (see e.g. [5, 25, 27]). Motivated essentially by that connection, our main objective is to establish a new I–Bessel sampling expansion formula which will be presented in the next section, together with a set of corresponding expansion results for I_ν, I_ν^2 and for the generalized hypergeometric function ${}_2F_3$, where the sampling reproduction kernel consists of the Fox-Wright generalized hypergeometric function ${}_p\Psi_q^*$.

The results about truncation error upper bounds for J–Bessel sampling for the band–limited Hankel transform can be found in [8, 31]. Recent progress was also made by Knockaert [16] with respect to the J–Bessel truncation procedure and Jankov Maširević *et al.* [12] in the case of Y–Bessel sampling. Thus, the last section is devoted to establishing sharp truncation error upper bounds for a newly derived truncated sampling series of modified Bessel functions I_ν.

2 I–Bessel sampling expansions and Sturm–Liouville differential equation

The main aim of this section is to establish a new Bessel–sampling expansion formula for a function which possesses an integral representation in terms of the modified Bessel function of the first kind I_ν. The derivation is based on the Sturm–Liouville differential equation. After that, we apply the obtained expansion to derive another Bessel sampling formulae for $I_{\nu+1}/I_\nu$, $1/I_\nu$, I_ν^2 and for a generalized hypergeometric function ${}_2F_3$ as well.

Firstly, the modified Bessel function of the first kind I_ν of the order ν is a particular solution of the Bessel–type differential equation

$$x^2y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0, \qquad x \in (0, \infty)\,,$$

which can be presented in the Sturm–Liouville form:

$$-(xy'(x))' + \frac{\nu^2}{x}y(x) = -xy(x), \qquad x \in (0,\infty)\,.$$

This in turn implies [4] that $\sqrt{x}I_\nu(x\sqrt{\lambda})$ satisfies the Sturm–Liouville differential equation

$$y''(x) - (\nu^2 - 1/4)x^{-2}y(x) = \lambda y(x), \qquad x \in (0,\infty)\,.$$

We notice that this is in fact a singular Sturm–Liouville problem.

In order to state our next auxiliary result, which we require to perform our results in this section, we mention some preliminary facts. In [26, p. 581], Zayed stated that if $\phi(x) = \phi(x,\lambda)$ and $\theta(x) = \theta(x,\lambda)$ are the solutions of the singular Sturm–Liouville boundary value problem such that

$$\begin{aligned}
&\phi(0) = \sin\alpha, \quad \phi'(0) = -\cos\alpha\,,\\
&\theta(0) = \cos\alpha, \quad \theta'(0) = \sin\alpha\,,
\end{aligned}$$

then it is known [23] that there exists a complex valued function m such that for every nonreal λ the appropriate Sturm–Liouville differential equation has a solution

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda) \in L^2(0,\infty)\,. \tag{1}$$

Throughout this section m will denote a meromorphic function that is real–valued on the real axis and whose singularities are simple poles on $\mathbb{R}$. The poles of m will be denoted by $\{\lambda_k\}_{k\in\mathbb{N}_0}$.

Theorem A. [26, p. 582, Theorem 3.1] Consider the singular Sturm–Liouville problem

$$\begin{aligned}
&y'' - q(x)y = -\lambda y, \qquad x \in [0,\infty),\\
&y(0)\cos\alpha = -y'(0)\sin\alpha\,,
\end{aligned}$$

where $q(x) \in \mathrm{C}[0,\infty)$. Assume that m is a meromorphic function that is real–valued on the real axis and whose only singularities are simple poles $\{\lambda_k\}_{k\in\mathbb{N}_0}$ on the non-negative real axis, and λ_0 will be reserved for the eigenvalue zero.

Let p be the smallest integer for which the series $\sum_{k\ge1}(\lambda_k)^{-p-1}$ converges.

(a) If none of λ_k is zero, set

$$G(\lambda) = \begin{cases} \displaystyle\prod_{k\ge0}\left(1-\frac{\lambda}{\lambda_k}\right)\exp\sum_{j=1}^{p}\frac{1}{j}\left(\frac{\lambda}{\lambda_k}\right)^j, & p\in\mathbb{N} \\ \displaystyle\prod_{k\ge0}\left(1-\frac{\lambda}{\lambda_k}\right), & p=0 \end{cases};$$

(b) If one λ_k is zero, say $\lambda_0 = 0$, set

$$G(\lambda) = \begin{cases} \lambda \prod_{k\geq 1}\left(1-\dfrac{\lambda}{\lambda_k}\right)\exp\sum_{j=1}^{p}\dfrac{1}{j}\left(\dfrac{\lambda}{\lambda_k}\right)^j, & p\in\mathbb{N} \\ \lambda \prod_{k\geq 1}\left(1-\dfrac{\lambda}{\lambda_k}\right), & p=0 \end{cases}.$$

Let $\Phi(x,\lambda) = G(\lambda)\psi(x,\lambda)$, $g(x) \in L^2(0,\infty)$ and

$$f(\lambda) = \int_0^\infty g(x)\,\Phi(x,\lambda)\,\mathrm{d}x.$$

Then f is an entire function that admits the sampling representation

$$f(\lambda) = \sum_{k\geq 0} f(\lambda_k)\frac{G(\lambda)}{(\lambda-\lambda_k)\,G'(\lambda_k)},$$

where the series converges uniformly on compact subsets of the complex λ–plane.

Now, we establish our main result in this section.

Theorem 1. *If for some $g \in L^2(0,a), a > 0$, the function F has an integral representation*

$$F(\lambda) = \frac{2^\nu\,\Gamma(\nu+1)}{\lambda^{\frac{\nu}{2}}\,a^{\nu+\frac{1}{2}}}\int_0^a g(x)\sqrt{x}I_\nu(x\sqrt{\lambda})\,\mathrm{d}x, \tag{2}$$

then the following sampling representation holds

$$F(\lambda) = \frac{2I_\nu(a\sqrt{\lambda})}{a\lambda^{\frac{\nu}{2}}}\sum_{k\geq 1}\frac{\lambda_k^{(\nu+1)/2}F(\lambda_k)}{(\lambda-\lambda_k)I'_\nu(a\sqrt{\lambda_k})}, \tag{3}$$

where $\lambda_k = -a^{-2}j_{\nu,k}^2$, $k\in\mathbb{N}$; $\nu > -1$ and the series converges uniformly on compact subsets of the complex λ–plane.

Moreover, let $g \in L^2(0,1)$ and assume that a function f possesses an integral expression, which reads as follows

$$f(t) = \int_0^1 g(x)\sqrt{x}I_\nu(tx)\,\mathrm{d}x. \tag{4}$$

Then the related sampling representation is

$$f(t) = 2I_\nu(t)\sum_{k\geq 1}\frac{t_k\,f(t_k)}{I_{\nu+1}(t_k)\left(t^2-t_k^2\right)}, \tag{5}$$

where $\nu > -1$ and $t_k = -\mathrm{i}\,j_{\nu,k}$ is the kth zero of I_ν.

Proof. In order to derive summation formula (3) we set $\phi(x,\lambda)$, $\theta(x,\lambda)$ and $\mu(\lambda)$ as

$$\begin{aligned}\phi(x,\lambda) &= \sqrt{ax}\left(I_\nu(a\sqrt{\lambda})\,K_\nu(x\sqrt{\lambda}) - I_\nu(x\sqrt{\lambda})\,K_\nu(a\sqrt{\lambda})\right)\\ \theta(x,\lambda) &= \sqrt{a\lambda x}\left(I'_\nu(a\sqrt{\lambda})\,K_\nu(x\sqrt{\lambda}) - I_\nu(x\sqrt{\lambda})\,K'_\nu(a\sqrt{\lambda})\right) + \frac{\phi(x,\lambda)}{2a}\\ m(\lambda) &= -\sqrt{\lambda}\,\frac{I'_\nu(a\sqrt{\lambda})}{I_\nu(a\sqrt{\lambda})} - \frac{1}{2a}\,.\end{aligned}$$

Now, from (1) we have that

$$\psi(x,\lambda) = \sqrt{a\lambda x}\left(\frac{I'_\nu(a\sqrt{\lambda})\,I_\nu(x\sqrt{\lambda})\,K_\nu(a\sqrt{\lambda})}{I_\nu(a\sqrt{\lambda})} - I_\nu(x\sqrt{\lambda})\,K'_\nu(a\sqrt{\lambda})\right)$$

i.e.

$$I_\nu(a\sqrt{\lambda})\,\psi(x,\lambda) = \sqrt{\frac{x}{a}}\,I_\nu(x\sqrt{\lambda})\,, \tag{6}$$

involving the Wronskian $\mathscr{W}[\cdot,\cdot]$ of the modified Bessel functions I_ν and K_ν [28, p. 80]

$$\mathscr{W}(K_\nu, I_\nu)(a\sqrt{\lambda}) = I'_\nu(a\sqrt{\lambda})\,K_\nu(a\sqrt{\lambda}) - I_\nu(a\sqrt{\lambda})\,K'_\nu(a\sqrt{\lambda}) = \frac{1}{a\sqrt{\lambda}}\,.$$

From the definition of m and the well–known identity $I_\nu(t) = \mathrm{i}^{-\nu} J_\nu(\mathrm{i}t)$, we find that $\lambda_k = -a^{-2}\, j_{\nu,k}^2$, $k \in \mathbb{N}$, where $j_{\nu,k}$ is the kth positive real zero of the Bessel function J_ν. Let us also mention that the zeros $j_{\nu,k}$, $k \in \mathbb{N}$ are positive real numbers for all $\nu > -1$ and there also holds [28, p. 479]

$$0 < j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \cdots\,.$$

Further, by Theorem A we conclude that

$$G(\lambda) = \prod_{k\geq 1}\left(1 + \frac{\lambda a^2}{j_{\nu,k}^2}\right)\,. \tag{7}$$

Now, with the help of the formula [28, p. 498]

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\prod_{k\geq 1}\left(1 - \frac{z^2}{j_{\nu,k}^2}\right), \qquad \Re\{\nu\} \notin \mathbb{Z}^-\,,$$

which by virtue of substitution $z \mapsto \mathrm{i}z$ becomes

$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\prod_{k\geq 1}\left(1 + \frac{z^2}{j_{\nu,k}^2}\right),$$

we can rewrite relation (7) as

$$G(\lambda) = \frac{2^\nu \Gamma(\nu+1) I_\nu(a\sqrt{\lambda})}{(a\sqrt{\lambda})^\nu}\,. \tag{8}$$

Now, from (6) and (8) we have

$$\Phi(x,\lambda) = G(\lambda)\,\psi(x,\lambda) = \frac{2^\nu \Gamma(\nu+1)}{(a\sqrt{\lambda})^\nu}\sqrt{\frac{x}{a}}\, I_\nu(x\sqrt{\lambda})\,.$$

The desired formula (3) readily follows by previous results and Theorem A.

Now, transforming integral representation (2) and the sum in (3) by taking $\lambda = t^2$, $\lambda_k = t_k^2$ and $a = 1$, being $I'_\nu(t) = I_{\nu+1}(t) + \frac{\nu}{t} I_\nu(t)$, we deduce that if for some $g \in L^2(0,1)$ the function F has an integral representation

$$F(t) = \frac{2^\nu\,\Gamma(\nu+1)}{t^\nu}\int_0^1 g(x)\sqrt{x} I_\nu(xt)\,\mathrm{d}x\,, \tag{9}$$

then the related sampling representation is

$$F(t) = \frac{2 I_\nu(t)}{t^\nu}\sum_{k\geq 1}\frac{t_k^{\nu+1}\,F(t_k)}{(t^2-t_k^2) I_{\nu+1}(t_k)}\,, \tag{10}$$

where $\nu > -1$ and $t_k = -\mathrm{i}\, j_{\nu,k}$ is the kth zero of $I_\nu(t)$.

Equivalently, if $f(t) := \dfrac{t^\nu F(t)}{2^\nu\Gamma(\nu+1)}$, from formulas (9) and (10) we can immediately deduce that if the function f has an integral representation (4), then the appropriate sampling representation is given by (5). □

Remark 1. Zayed [26, p. 592] obtained summation formulae analogous to (3) and (5) for the Bessel function of the first kind J_ν.

Also, a special case of the sampling representation formula (3), when $a = 1$, was derived by Ismail and Kelker (see [11, Theorem 6.4, p. 899]), where they assumed that F is a single–valued entire function with the asymptotic behavior $F(\lambda) = O(\lambda^{-\nu/2-1/2}\mathrm{e}^{\sqrt{\lambda}})$, as $|\lambda| \to \infty$ uniformly in every sector $|\arg\lambda| \leq \pi - \varepsilon,\ 0 < \varepsilon < \pi$.

Now, we present three summation formulae for a modified Bessel function I_ν.

Corollary 1.1. *For $\nu > -1$ we have*

$$\frac{I_{\nu+1}(t)}{I_\nu(t)} = 2t\sum_{k\geq 1}\frac{1}{t^2-t_k^2} = 2t\sum_{k\geq 1}\frac{1}{t^2+j_{\nu,k}^2}\,. \tag{11}$$

Moreover, there holds

$$\pi\coth(\pi t) = 2t\sum_{k\geq 1}\frac{1}{t^2+k^2} + \frac{1}{t}, \qquad t \neq 0\,.$$

Proof. By rewriting the integral expression [6, p. 668, Eq. 6.561.7]

$$t^{-1}I_{\nu+1}(t) = \int_0^1 x^{\nu+1} I_\nu(tx)\,\mathrm{d}x, \qquad \nu > -1\,,$$

into

$$t^{-1}I_{\nu+1}(t) = \int_0^1 x^{\nu+\frac{1}{2}} \sqrt{x} I_\nu(tx)\,\mathrm{d}x\,,$$

we recognize that

$$g(x) = x^{\nu+1/2} \in L^2(0,1), \text{ for all } \nu > -1 \qquad \text{and} \qquad f(t) = t^{-1}I_{\nu+1}(t)\,.$$

Now, from (5) we can immediately get (11). Using the well–known identities

$$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\frac{\sinh z}{\sqrt{z}}, \qquad I_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi}}\frac{z\cosh z - \sinh z}{z^{3/2}}$$

and bearing in mind that zeros of $J_{\frac{1}{2}}$ are of the form $j_{\frac{1}{2},k} = k\pi$, $k \in \mathbb{N}$, for $\nu = 1/2$ equation (11) becomes

$$\frac{\cosh t}{\sinh t} - \frac{1}{t} = 2t\sum_{k\geq 1}\frac{1}{t^2+(k\pi)^2}, \qquad t \neq 0$$

and this expression is equivalent to the hyperbolic cotangent sum. □

Remark 2. Equality (11) is already known as a Mittag–Leffler expansion [3, Eq. 7.9.3].

The formula

$$\sum_{k\geq 1}\frac{1}{t^2+k^2} = \frac{\pi}{2t}\coth(\pi t) - \frac{1}{2t^2}, \qquad t \neq 0$$

was considered by Hamburger [7, p. 130, Eq. (C)] in a slightly different form

$$1 + 2\sum_{k\geq 1}\mathrm{e}^{-2\pi kt} = \mathrm{i}\cot\pi\mathrm{i}t = \frac{1}{\pi t} + \frac{2t}{\pi}\sum_{k\geq 1}\frac{1}{t^2+k^2}, \qquad t \neq \mathrm{i}k\,.$$

Also, subsequent complex analytical generalizations of Hamburger's formula can be found in [2].

Corollary 1.2. *For $\nu \in (-1,1)\backslash\{0\}$ it holds*

$$\frac{1}{I_\nu(t)} = \frac{2^\nu\Gamma(\nu)}{t^{\nu-1}}\left(\frac{\nu}{t} - \frac{t}{2^{\nu-1}\Gamma(\nu)}\sum_{k\geq 1}\frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})\,(t^2+j_{\nu,k}^2)}\right)\,. \tag{12}$$

Proof. From the recursive relation

$$t\,I_{\nu-1}(t) - t\,I_{\nu+1}(t) = 2\nu\,I_\nu(t)$$

and equality (11) we can conclude that

$$\frac{I_{\nu-1}(t)}{I_\nu(t)} = \frac{I_{\nu+1}(t)}{I_\nu(t)} + \frac{2\nu}{t} = 2t\sum_{k\geq 1}\frac{1}{t^2+j_{\nu,k}^2} + \frac{2\nu}{t}, \quad \nu > -1\,. \tag{13}$$

Now, by using the integral expression [6, p. 668, Eq. 6.561.11]

$$t^{-1}I_{\nu-1}(t) - \frac{t^{\nu-2}}{2^{\nu-1}\Gamma(\nu)} = \int_0^1 x^{1-\nu}\, I_\nu(tx)\,\mathrm{d}x\,,$$

where we recognize that $g(x) = x^{1/2-\nu} \in L^2(0,1)$ for all $\nu < 1$ and $f(t) = t^{-1}I_{\nu-1}(t) - \frac{t^{\nu-2}}{2^{\nu-1}\Gamma(\nu)}$, from (5) we can conclude that

$$\frac{I_{\nu-1}(t)}{I_\nu(t)} - \frac{t^{\nu-1}}{2^{\nu-1}\Gamma(\nu)\,I_\nu(t)} = 2t\sum_{k\geq 1}\frac{1}{t^2+j_{\nu,k}^2}\left(1 - \frac{t_k^{\nu-1}}{2^{\nu-1}\Gamma(\nu)\,I_{\nu+1}(t_k)}\right).$$

Combining the previous expression and (13) we get

$$\frac{t^{\nu-1}}{2^{\nu-1}\Gamma(\nu)\,I_\nu(t)} = \frac{2\nu}{t} + \frac{2t}{2^{\nu-1}\Gamma(\nu)}\sum_{k\geq 1}\frac{t_k^{\nu-1}}{I_{\nu+1}(t_k)\,(t^2+j_{\nu,k}^2)}\,,$$

which immediately gives the desired summation formula (12). Here, we also assumed that $\nu \neq 0$, because $\Gamma(0) = (-1)! = +\infty$. □

Remark 3. A result similar to (12) was deduced by Ismail and Kelker (see [11, Theorem 4.10, p. 896]). They proved that

$$\frac{t^{\nu/2}}{I_\nu(\sqrt{t})} = -2\sum_{k\geq 1}\frac{j_{\nu,k}^{\nu+1}}{(t+j_{\nu,k}^2)J_\nu'(j_{\nu,k})},\ \nu > -1.$$

Corollary 1.3. *For $\nu > 0$ we have*

$$I_\nu^2\left(\frac{t}{2}\right) = 2(-1)^{-\nu}\,t^\nu\,I_\nu(t)\sum_{k\geq 1}\frac{j_{\nu,k}^{1-\nu}I_\nu^2\left(\frac{\mathrm{i}}{2}\,j_{\nu,k}\right)}{(t^2+j_{\nu,k}^2)\,J_{\nu+1}(j_{\nu,k})}\,. \tag{14}$$

Proof. Using the same procedure as above, with the help of the integral representation [6, p. 672, Eq. 6.567.12]

$$2^{-\nu-1}\sqrt{\pi}t^{-\nu}\Gamma\left(\nu+\tfrac{1}{2}\right)I_\nu^2\left(\tfrac{t}{2}\right) = \int_0^1 x^\nu(1-x^2)^{\nu-1/2}I_\nu(tx)\,\mathrm{d}x\,,$$

where the kernel function $g(x) = (x-x^3)^{\nu-1/2}$ is in $L^2(0,1)$ for all $\nu > 0$, setting $f(t) = 2^{-\nu-1}\sqrt{\pi}t^{-\nu}\Gamma\left(\nu+\frac{1}{2}\right)I_\nu^2\left(\frac{t}{2}\right)$, by virtue of (5) and using the identities $I_\nu(z) = \mathrm{i}^{-\nu}J_\nu(\mathrm{i}z)$, $I_\nu(-z) = (-1)^\nu I_\nu(z)$ we arrive at (14). □

Finally, by using Theorem 1, we derive the sampling expansion formula for a generalized hypergeometric function ${}_2F_3$. Firstly, the *generalized hypergeometric function* ${}_pF_q[z]$ with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ is defined as the series [20]

$$ {}_pF_q[z] = {}_pF_q\left[\begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array} \middle| z \right] = \sum_{n\geq 0} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!}, $$

where $(a)_n$ denotes the Pochhammer symbol (or the shifted factorial) [19]

$$ (a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1). $$

When $p \leq q$, the generalized hypergeometric function converges for all complex values of z; thus, ${}_pF_q[z]$ is an entire function. When $p > q+1$, the series converges only for $z = 0$, unless it terminates (as when one of the parameters a_i is a negative integer) and in that case it is just a polynomial in z. When $p = q+1$, the series converges in the unit disk $|z| < 1$, and also for $|z| = 1$ provided that $\Re\left\{\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right\} > 0$.

Further, we need the Fox-Wright generalized hypergeometric function ${}_p\Psi_q^*[\cdot]$ with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$, which is defined by [15, p. 56]

$$ {}_p\Psi_q^*\left[\begin{array}{c} (a_1,\rho_1), \cdots, (a_p,\rho_p) \\ (b_1,\sigma_1), \cdots, (b_q,\sigma_q) \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{\rho_j n}}{\prod_{j=1}^{q} (b_j)_{\sigma_j n}} \frac{z^n}{n!}, \tag{15} $$

where $a_j, b_k \in \mathbb{C}$ and $\rho_j, \sigma_k \in \mathbb{R}_+$, $j = 1, \cdots, p$; $k = 1, \cdots, q$. The defining series in (15) converges in the whole complex z-plane when

$$ \Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j > -1\,; $$

when $\Delta = 0$, the series in (15) converges for $|z| < \nabla$, where

$$ \nabla := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j} \right) \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j} \right). $$

Corollary 1.4. *For all* t, λ, ν, μ *such that* $\min\left(t, \lambda-1, \nu+1, \mu-\frac{1}{2}\right) > 0$ *we have*

$$ {}_2F_3\left[\begin{array}{c} \frac{1}{2}(\nu+\lambda), \quad \frac{1}{2}(\nu+\lambda+1) \\ \nu+1, \frac{1}{2}(\nu+\lambda+\mu), \frac{1}{2}(\nu+\lambda+\mu+1) \end{array} \middle| \frac{t^2}{4} \right] $$
$$ = \frac{2I_\nu(t)}{t^\nu} \sum_{k\geq 1} \frac{j_{\nu,k}^{\nu+1} \, {}_1\Psi_2^*\left[\begin{array}{c} (\nu+\lambda, 2) \\ (\nu+1,1), (\nu+\lambda+\mu, 2) \end{array} \middle| \frac{-j_{\nu,k}^2}{4} \right]}{(t^2+j_{\nu,k}^2) J_{\nu+1}(j_{\nu,k})}. \tag{16} $$

Proof. Consider the integral representation formula [6, p. 673, 6.569] derived for J_ν. Its corresponding modified Bessel I_ν–variant reads as follows:

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1} I_\nu(tx)\,\mathrm{d}x = \frac{2^{1-2\nu-\lambda-\mu}\sqrt{\pi}\,t^\nu\,\Gamma(\nu+\lambda)\,\Gamma(\mu)}{\Gamma(\nu+1)\,\Gamma\left(\frac{\nu+\lambda+\mu}{2}\right)\,\Gamma\left(\frac{\nu+\lambda+\mu+1}{2}\right)} \times {}_2F_3\left[\begin{array}{c} \frac12(\nu+\lambda),\quad \frac12(\nu+\lambda+1) \\ \nu+1,\, \frac12(\nu+\lambda+\mu),\, \frac12(\nu+\lambda+\mu+1)\end{array}\Big|\, \frac{t^2}{4}\right],$$

and it is valid for $\min(t,\lambda,\nu+\lambda,\mu)>0$. Choosing $g(x)=x^{\lambda-\frac32}(1-x)^{\mu-1}\in L^2(0,1)$ for $\mu>\frac12$ and $\lambda>1$ and then applying Theorem 1 we arrive at

$$t^\nu\,{}_2F_3\left[\begin{array}{c} \frac12(\nu+\lambda),\quad \frac12(\nu+\lambda+1) \\ \nu+1,\, \frac12(\nu+\lambda+\mu),\, \frac12(\nu+\lambda+\mu+1)\end{array}\Big|\, \frac{t^2}{4}\right] = 2I_\nu(t)\sum_{k\ge1}\frac{j_{\nu,k}^{\nu+1}\,{}_2F_3\left[\begin{array}{c} \frac{\nu+\lambda}{2},\quad \frac{\nu+\lambda+1}{2} \\ \nu+1,\, \frac{\nu+\lambda+\mu}{2},\, \frac{\nu+\lambda+\mu+1}{2}\end{array}\Big|\, \frac{-j_{\nu,k}^2}{4}\right]}{(t^2+j_{\nu,k}^2)\,J_{\nu+1}(j_{\nu,k})}. \tag{17}$$

Now, with the aid of the property of the Pocchammer symbol

$$(x)_{2n} = 2^{2n}\left(\frac{x}{2}\right)_n\left(\frac{1+x}{2}\right)_n,$$

we have that

$$\begin{aligned}{}_2F_3&\left[\begin{array}{c} \frac{\nu+\lambda}{2},\quad \frac{\nu+\lambda+1}{2} \\ \nu+1,\, \frac{\nu+\lambda+\mu}{2},\, \frac{\nu+\lambda+\mu+1}{2}\end{array}\Big|\, \frac{-j_{\nu,k}^2}{4}\right] \\ &= \sum_{n\ge0}\frac{(\nu+\lambda)_{2n}}{(\nu+1)_n(\nu+\lambda+\mu)_{2n}}\frac{(-j_{\nu,k}^2)^n}{4^n\,n!} \\ &= {}_1\Psi_2^*\left[\begin{array}{c} (\nu+\lambda,2) \\ (\nu+1,1),(\nu+\lambda+\mu,2)\end{array}\Big|\, \frac{-j_{\nu,k}^2}{4}\right].\end{aligned} \tag{18}$$

Summing (17) and (18) we obtain the summation formula (16). □

3 Truncation error upper bounds in I–Bessel sampling expansions

In this section our aim is to derive a uniform upper bound for the truncation error for the Bessel–sampling expansion (5).

The truncated sampling reconstruction sum of the size $N\in\mathbb{N}$ for the Bessel–sampling formula (5) is defined as

$$\mathscr{S}_N^I(f;t) = 2I_\nu(t)\sum_{k=1}^N\frac{t_k\,f(t_k)}{I_{\nu+1}(t_k)\left(t^2-t_k^2\right)},$$

where $t \in \mathbb{R}$, $t_k = -\mathrm{i}\, j_{\nu,k}$ is the kth zero of I_ν, $\nu > -1$ and the function f has a band–region contained in $(0,1)$. Let us also define the truncation error of the order N as the quantity

$$\mathscr{T}_N^I(f;t) = \left|f(t) - \mathscr{S}_N^I(f;t)\right| = \left|2I_\nu(t) \sum_{k\geq N+1} \frac{t_k f(t_k)}{I_{\nu+1}(t_k)\left(t^2 - t_k^2\right)}\right|.$$

We are looking for an upper bound for the truncation error $\mathscr{T}_N^I(f;t)$ in the case when the input function possesses a polynomially decaying upper bound like

$$|f(t)| \leq A\,|t|^{-(r+1)}, \qquad A > 0,\, r > 0\,, t \neq 0.$$

Thus, for all $\nu > -1$ we have

$$\mathscr{T}_N^I(f;t) \leq 2A \sum_{k\geq N+1} \frac{|I_\nu(t)|}{j_{\nu,k}^r(t^2 + j_{\nu,k}^2)\,\left|J_{\nu+1}(j_{\nu,k})\right|}\,,$$

because of the identity $I_\nu(t) = \mathrm{i}^{-\nu} J_\nu(\mathrm{i}t)$ and the fact that all zeros $j_{\nu,k}$ are positive for $\nu > -1$.

Using an integral representation [28, p. 181, Eq. (4)]

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi \mathrm{e}^{z\cos t}\cos \nu t\,\mathrm{d}t - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty \mathrm{e}^{-z\cosh t - \nu t}\,\mathrm{d}t, \quad \nu > 0$$

we can conclude that

$$|I_\nu(t)| \leq I_0(t) + \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-t\cosh x}\,\mathrm{d}x = I_0(t) + \frac{1}{\pi}K_0(t), \qquad t > 0,$$

thus

$$\sup_{\nu<t<y_{\nu,2}} |I_\nu(t)| = I_0(y_{\nu,2}) + \frac{1}{\pi}K_0(\nu) := H_1\,. \tag{19}$$

Using (19) and the particular value of the Rayleigh function [28, p. 502]

$$\sigma_\nu^{(r)} = \sum_{k\geq 1} \frac{1}{j_{\nu,k}^{2r}}, \qquad r \in \mathbb{N}\,,$$

for $r = 1$, that is $\sigma_\nu^{(1)} = (4(\nu+1))^{-1}$ bearing in mind that $|t| > 0$, it holds

$$\begin{aligned}\mathscr{T}_N^I(f;t) &< \frac{2A\,H_1}{\min\limits_{k\geq N+1} j_{\nu,k}^r\,|J_{\nu+1}(j_{\nu,k})|}\sum_{k\geq 1}\frac{1}{j_{\nu,k}^2} \\ &= \frac{A\,H_1}{2(\nu+1)\min\limits_{k\geq N+1} j_{\nu,k}^r\,|J_{\nu+1}(j_{\nu,k})|}\,.\end{aligned} \tag{20}$$

It remains to minimize the expression in the denominator of (20). For that purpose we exploit Krasikov's bound [18, p. 84, Theorem 2]:

$$J_\nu^2(x) \geq \frac{4\big(x^2 - (2\nu+1)(2\nu+5)\big)}{\pi\big((4x-\nu)^{\frac{3}{2}} + \mu\big)}, \qquad x > \frac{1}{2}\sqrt{\mu + \mu^{\frac{3}{2}}}\,,$$

where

$$\mu = (2\nu+1)(2\nu+3), \qquad \nu > -\tfrac{1}{2}\,.$$

In [18] Krasikov pointed out that this lower bound is poor in the transition region around zeros $j_{\nu,k}$, while it fits well the Bessel function of the first kind $J_\nu(t)$ in the oscillatory region. Since we have to estimate $J_{\nu+1}(j_{\nu,k})$, these values are obviously separated from zero as $j_{\nu+1,k}$ and $j_{\nu,k}$ interlace and the latter zero belongs to the oscillatory region of $J_{\nu+1}(t)$. Hence

$$\big|J_{\nu+1}(j_{\nu,k})\big| \geq \frac{2}{\sqrt{\pi}} \left\{ \frac{j_{\nu,k}^2-(2\nu+3)(2\nu+7)}{(4j_{\nu,k}-\nu-1)^{\frac{3}{2}}+\mu^*} \right\}^{\frac{1}{2}}\,, \tag{21}$$

where

$$\mu^* = \frac{2\nu+5}{2\nu+1}\,\mu > 15.$$

The range of validity of (21) is

$$x = j_{\nu,N+1} > \frac{1}{2}\sqrt{\mu^*+(\mu^*)^{\frac{3}{2}}} \approx 4.27447\,. \tag{22}$$

Thus, for N large enough, applying the MacMahon asymptotics for the zeros of the cylinder functions [28, p. 506] (see also Schläfli's footnote [21, p. 137])

$$y_{\nu,N} = \big(N+\tfrac{\nu}{2}-\tfrac{1}{4}\big)\pi + \mathscr{O}(N^{-1}), \qquad N\to\infty, \tag{23}$$

and the well–known interlacing inequalities for the positive zeros $j_{\nu,k}, j'_{\nu,k}, y_{\nu,k}$ and $y'_{\nu,k}$ of Bessel functions $J_\nu(t), J'_\nu(t)$, $Y_\nu(t)$ and $Y'_\nu(t)$, respectively [1, p. 370],

$$\nu \leq j'_{\nu,1} < y_{\nu,1} < y'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < y_{\nu,2} < \cdots,$$

we have that the solution of (22) in N for the range $\nu>0$ becomes:

$$N + \mathscr{O}(N^{-1}) > \frac{1}{2\pi}\sqrt{15+15^{\frac{3}{2}}} + \frac{1-2\nu}{4} \approx 1.61061 - \frac{\nu}{2}.$$

Thus, (22) is not redundant for

$$\nu \leq \frac{1}{\pi}\sqrt{15+15^{\frac{3}{2}}} - \frac{3}{2} =: \nu^* \approx 1.22141\,.$$

Now, bearing in mind that $\nu\in(0,\nu^*]$, by (21) we deduce

$$j_{\nu,k}^r\,|J_{\nu+1}(j_{\nu,k})| \geq \frac{2}{\sqrt{\pi}}\, j_{\nu,k}^r \left\{ \frac{j_{\nu,k}^2-(2\nu+3)(2\nu+7)}{(4j_{\nu,k}-\nu-1)^{\frac{3}{2}}+\mu^*} \right\}^{\frac{1}{2}} =: L_k(\nu).$$

It is not hard to see that the function

$$x \mapsto x^r \left\{ \frac{x^2-(2\nu+3)(2\nu+7)}{(4x-\nu-1)^{\frac{3}{2}}+\mu^*} \right\}^{\frac{1}{2}}$$

monotonically increases in its domain, thus

$$\min_{k\geq N+1} j_{\nu,k}^{r}\, |J_{\nu+1}(j_{\nu,k})| \geq L_{N+1}(\nu)\,,$$

where we assume $N \geq 2$, because of positivity of the expression in the numerator of $L_{N+1}^2(\nu)$. Thus, we proved the result given in the following theorem.

Theorem 2. *Let $\nu \in (0, \nu^*]$, where*

$$\nu^* = \frac{1}{\pi}\sqrt{15+15^{\frac{3}{2}}} - \frac{3}{2}.$$

Then for all $t \in (\nu, y_{\nu,2})$, $\min(A,r) > 0$ and all $N \geq 3$ there holds the truncation error upper bound

$$\mathscr{T}_N^I(f;t) < \frac{A\,H_1}{2(\nu+1)\,L_N(\nu)} := U_N^I(t), \tag{24}$$

where

$$H_1 = I_0(y_{\nu,2}) + \frac{1}{\pi}K_0(\nu),$$

$$L_N(\nu) = \frac{2}{\sqrt{\pi}}\, j_{\nu,N}^{r} \left\{ \frac{j_{\nu,N}^2 - (2\nu+3)(2\nu+7)}{(4j_{\nu,N}-\nu-1)^{\frac{3}{2}} + \mu^*} \right\}^{\frac{1}{2}}.$$

Moreover, for N large enough the asymptotics of the truncation error is

$$\mathscr{T}_N^I(f;t) = \mathscr{O}\left(N^{-r-\frac{1}{4}}\right).$$

Proof. As already proved, an upper bound (24), it remains to show the asymptotics of the truncation error $\mathscr{T}_N^I(f;t)$. Thus, for fixed t and N large enough, again by applying (23) we have

$$\mathscr{T}_N^I(f;t) = \mathscr{O}\left(U_N^I(t)\right) = \mathscr{O}\left(\frac{1}{L_N(\nu)}\right) = \mathscr{O}\left((j_{\nu,N})^{-r-\frac{1}{4}}\right) = \mathscr{O}\left(N^{-r-\frac{1}{4}}\right),$$

which completes the proof. □

In addition, we will consider an example which includes the results obtained in Corollary 1.3 to demonstrate the Bessel–sampling approximation behavior.

Example 1. Let us denote

$$h(t) = \frac{(-1)^\nu I_\nu^2\left(\frac{t}{2}\right)}{2t^\nu I_\nu(t)}, \qquad \mathscr{S}_N^I(h;t) = \sum_{k\geq 1} \frac{j_{\nu,k}^{1-\nu} I_\nu^2\left(\frac{\mathrm{i}}{2}\, j_{\nu,k}\right)}{(t^2+j_{\nu,k}^2)\,J_{\nu+1}(j_{\nu,k})}.$$

In Fig. 1 we present the input function h and the truncated sampling I–Bessel sampling approximation sums $\mathscr{S}_N^I(h;t)$ for $N = 15, 150, 3000$, respectively, on the t–domain $[0, j_{0,1}] \approx [0, 2.40483]$ in case $\nu = 0$.

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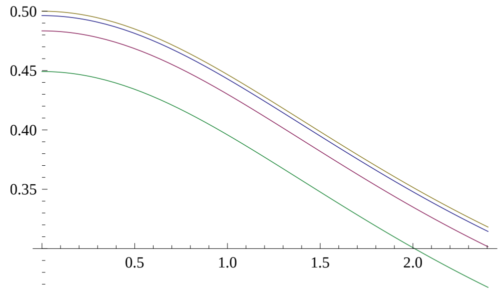

Figure 1
I–Bessel–sampling approximation patterns associated with Eq. (14) in Corollary 1.3. Legend: $h(t)$ – yellow, $\mathscr{S}^I_{15}(h;t)$ – green, $\mathscr{S}^I_{150}(h;t)$ – violet and $\mathscr{S}^I_{3000}(h;t)$ – blue.

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$$\overset{a}{J}(\beta_1 x), \overset{a}{J}(\beta_2 x), \overset{a}{J}(\beta_3 x), \cdots\cdots$$

wo $\beta_1, \beta_2, \beta_3, \ldots$ die positiven Wurzeln der Gleichung $\overset{a}{J}(\beta) = 0$ vorstellen, *Math. Ann.*, **10** (1876), 137–142.

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