Testing of Markov Assumptions Based on the Dynamic Specification Test

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Abstract: An alternative approach to testing linearity against Markov-switching type non-linearity is proposed. The main problem of the classical testing via the likelihood ratio test is that the test statistic does not have a standard distribution. Therefore, time-consuming simulations must be carried out. Instead of the classical test we suggest using Hamilton's dynamic specification test for the validity of Markov assumptions. We show that this new approach provides much faster calculations. With the same idea we calculate the test for remaining non-linearity to compare 2-regime with 3-regime models. We compare these two approaches with 100 selected real time series from economy and finance.

Keywords: Markov-switching model; Markov assumptions; testing linearity, dynamic specification test

1 Introduction

Markov-switching models have achieved a great expansion in non-linear time series modeling because of their great descriptive properties. The idea is that model parameters can acquire different values. This depends on the "regime" or "state" the model is in. The parameter switching follows the dynamic behavior of economic and financial time series quite well. For instance, one regime can express an expansion and the second one a recession. Regime changes are caused by dramatic, occasional and rare events like wars, political instability, financial crises and so on. Such discrete shifts in parameters can cause changes in the expected value, the variance or coefficients of the model.

In 1993, Granger [1] described a procedure which should be kept in case of a non-linear modeling. We accept the principle from the specific to the more common. So we start with a simpler linear model and then after maintaining given conditions we go to more complex non-linear models. Here are the steps for such modeling:

1 Choose an appropriate linear model AR(q) for the examined time series,

2 test the null hypothesis about model linearity against non-linearity (if the null hypothesis is rejected then we continue to the next step),

3 estimate the parameters of chosen non-linear model,

4 check the appropriateness of the chosen non-linear model by diagnostic tests,

5 modify the model if necessary and

6 use the model for the description and prediction of the time series.

In the following sections I briefly summarize the basis of Markov-switching models and the classical testing linearity against Markov-switching type non-linearity via simulations. Then I introduce the main ideas about the proposed testing, which is suggested as a faster alternative to the old one. In the end I present my results in comparing both approaches.

2 Markov-Switching Models

Markov-switching models (MSW models) belong to regime-switching models whose regime is determined by an unobservable variable. This means that we cannot determine in which regime the process is exactly but only with some probability.

Suppose that the regime occurring in time t is described by a random variable s_t and if we distinguish N possible regimes, the random variable s_t can attain values from the set $\{1,2,...,N\}$. We can define the stochastic process $\{s_t\}$, which is a sequence of random variables s_t. In 1989, Hamilton [2] proposed to specify this stochastic process as a first-order Markov process. This means that the process has to satisfy this property:

$$P(s_t = j \mid s_{t-1} = i, s_{t-2} = k, ...) = P(s_t = j \mid s_{t-1} = i) = p_{ij}. \quad (1)$$

Thus, the regime in time t depends only on the previous regime in time t-1. Such a described process is called an N-state Markov chain and $\{p_{ij}\}_{i,j=1,2,...,N}$ are called transitional probabilities. They represent the probability of change that the process in the regime i in time t-1 is followed by the regime j in time t. Therefore, it holds that

$$p_{i1} + p_{i2} + ... + p_{iN} = 1 \quad \text{for } i = 1,2,...,N. \quad (2)$$

In this paper we suppose the following MSW model form:

$$y_t = \phi_{0,s_t} + \phi_{1,s_t} y_{t-1} + ... + \phi_{q_{s_t},s_t} y_{t-q} + \varepsilon_t \text{ for } t = q_{s_t} + 1,...,T \text{ and } s_t = 1,2,...,N. \quad (3)$$

It is described by the AR(q) model in particular regimes, where y_t is the t^{th} observation, q is a model order ($q = \max\{q_1, q_2, ..., q_N\}$), T is the length of time series and ε_t is the i.i.d. white noise distributed with $N(0,\sigma_\varepsilon^2)$.

In this case the MSW model density of y_t conditional on the random variable s_t and the history of observations has the follow-up form

$$f(y_t \mid s_t = j, \Omega_{t-1}; \theta) = \frac{1}{\sqrt{2\pi}\sigma_\varepsilon} \exp\left\{ \frac{-(y_t - \varphi_j' \mathbf{x_t})^2}{2\sigma_\varepsilon^2} \right\}, \quad (4)$$

where $\varphi_\mathbf{j} = (\phi_{0,j}, \phi_{1,j}, ..., \phi_{q,j})'$ is a vector of AR coefficients for regime j, $\mathbf{x_t} = (1, y_{t-1}, ..., y_{t-q})'$ and $\Omega_{t-1} = (y_{t-1}, y_{t-2}, ..., y_1)$ is the history of observations.

3 The Classical Approach to Testing

As we mentioned in the introduction, one of the steps we should follow is testing linearity against the Markov-switching type non-linearity. Simply, we examine the suitability of a non-linear model instead of a linear model. The classical approach to such testing is the likelihood ratio test with a null hypothesis $H_0 : \varphi_1 = \varphi_2$ against its alternative $H_1 : \phi_{i,1} \neq \phi_{i,2}$ for at least one $i \in \{0,1,2,...,q\}$, where $\varphi_i = (\phi_{0,i}, \phi_{1,i}, ..., \phi_{q,i})'$ is a vector of AR coefficients for the i^{th} regime (i=1,2). The null hypothesis represents the linear model against the alternative hypothesis of the 2-regime MSW non-linear model. The likelihood ratio test statistic has the following form

$$LR = L_{MSW} - L_{AR}, \quad (5)$$

where L_{MSW} and L_{AR} are logarithms of likelihood functions, the first one for the suitable 2-regime MSW model and the second one for the best AR model.

One serious problem arises now: the problem of nuisance parameters. We should realize that in the case of the linear model we estimate a lower amount of parameters than in the case of the MSW model, where we have added transitional probabilities p_{ij} to the parameter vector. Hansen [6] proved in 1992 that this test statistic (5) has a non-standard probabilistic distribution. Such a distribution cannot be expressed analytically and for calculating critical values we need to carry out a simulation. The simulation is an experiment which consists of generating a large number (at least 5000) of artificial time series y_t^* according to the model representing the null hypothesis. The next step is to estimate the parameters of the best AR model and MSW model for each generated time series and to calculate the corresponding likelihood ratio statistic (5). Thus, we get critical values and then we are able to do the testing. It is necessary to do the simulation for each time series and each model order q distinctively.

The big disadvantage of this approach is the computation time. It takes “hours” to calculate a single simulation. Of course the computation time depends on the computer performance or the length of the time series. In the last section, using real data I show the difference between the computation times, comparing both approaches to testing.

Instead of such a time consuming test, we propose using the Newey-Tauchen-White test [7, 10, 11], the score function and the specification test for the validity of Markov assumptions, which was proposed in Hamilton [4]. We describe the new testing in the next section.

4 A New Approach to Testing Linearity

For this new approach we need a score function, the White test [11] for serial correlation, which uses conditional moment tests proposed by Newey [7] and Tauchen [10] and the dynamic specification test proposed by Hamilton [4].

4.1 Score Function

The score function for the t^{th} observation is defined as the vector of partial derivations of the logarithm of the conditional likelihood function with respect to the parameter vector θ

$$\mathbf{h_t}(\theta) \equiv \frac{\partial \log f(y_t \mid \Omega_{t-1}, \theta)}{\partial \theta}, \tag{6}$$

where y_t is the t^{th} observation and $\Omega_{t-1} = (y_{t-1}, y_{t-2}, ..., y_1)$ is a history of observations. We have to calculate the score for the whole length T of the time series.

Hamilton in [4] derived the score function for such a described MSW model. The score function has the form

$$\frac{\partial \ln f(y_1 \mid \Omega_0; \theta)}{\partial \alpha} = \sum_{s_1=1}^{N} \frac{\partial \ln f(y_1 \mid \mathbf{x_1}, s_1; \theta)}{\partial \alpha} P(s_1 \mid \Omega_1) \tag{7a}$$

for t=1 and

$$\frac{\partial \ln f(y_t \mid \Omega_{t-1};\theta)}{\partial \alpha} = \sum_{s_t=1}^{N} \frac{\partial \ln f(y_t \mid \mathbf{x_t}, s_t;\theta)}{\partial \alpha} P(s_t \mid \Omega_t) + \\ + \sum_{\tau=1}^{t-1}\sum_{s_\tau=1}^{N} \frac{\partial \ln f(y_\tau \mid \mathbf{x}_\tau, s_\tau;\theta)}{\partial \alpha}\{P(s_\tau \mid \Omega_t) - P(s_\tau \mid \Omega_{t-1})\} \tag{7b}$$

for t=2,3,…,T where the parameter vector consists of these elements - $\theta = (\alpha, \mathbf{p})$. The vector α includes AR coefficients for all regimes and model residual dispersion, which is $\alpha = (\varphi_1', \varphi_2', ..., \varphi_N', \sigma_\varepsilon^2)$. The vector $\mathbf{p}$ is a vector of transitional probabilities p_{ij} with omitting redundant parameters, which we are able to express by others as follows

$$p_{iN} = 1 - p_{i1} - p_{i2} - .... - p_{i,N-1}, \qquad i = 1,...,N.$$

Elements of the score function derived with respect to transitional probabilities have the form

$$\begin{aligned}\frac{\partial \ln f(y_t \mid \Omega_{t-1};\theta)}{\partial p_{ij}} &= p_{ij}^{-1}.P(s_t = j, s_{t-1} = i \mid \Omega_t) - p_{iN}^{-1}.P(s_t = N, s_{t-1} = i \mid \Omega_t) \\ &+ p_{ij}^{-1}\left\{\sum_{\tau=2}^{t-1}[P(s_\tau = j, s_{\tau-1} = i \mid \Omega_t) - P(s_\tau = j, s_{\tau-1} = i \mid \Omega_{t-1})]\right\} \\ &- p_{iN}^{-1}\left\{\sum_{\tau=2}^{t-1}[P(s_\tau = N, s_{\tau-1} = i \mid \Omega_t) - P(s_\tau = N, s_{\tau-1} = i \mid \Omega_{t-1})]\right\} \\ &+ \sum_{s_1=1}^{N} \frac{\partial \ln P(s_1;\mathbf{p})}{\partial p_{ij}}[P(s_1 \mid \Omega_t) - P(s_1 \mid \Omega_{t-1})]\end{aligned} \tag{8a}$$

for $i = 1,2,...,N,\ j = 1,2,...,N,\ t = 2,...,T$ and

$$\frac{\partial \ln f(y_1 \mid \Omega_0;\theta)}{\partial p_{ij}} = \sum_{s_1=1}^{N} \frac{\partial \ln P(s_1;\mathbf{p})}{\partial p_{ij}} P(s_1 \mid \Omega_1) \tag{8b}$$

for t=1. The calculation procedure and more details can be found in Hamilton [4].

4.2 The White Test and the Dynamic Specification Test

As mentioned above, White proposed a test for serial correlation by using conditional moment tests from Newey and Tauchen. For the construction of this test we need $(l \times 1)$ vector $\mathbf{c_t}(\hat{\theta})$, which is just a representative of examined properties from an outer product of the t^{th} score function and one-lagged, that is $(t-1)^{th}$, score function - $[\mathbf{h_t}(\hat{\theta})].[\mathbf{h_{t-1}}(\hat{\theta})]'$. This test is based on an assumption that if data are really generated from distribution $f(y_t \mid \Omega_{t-1}, \theta_0)$, where θ_0 is the vector of true parameters, then the follow-up equation $E(\mathbf{h_t}(\theta_0)\,|\,\Omega_{t-1}) = 0$ must be satisfied. This means that if the model is correctly specified, the score function $\mathbf{h_t}(\theta_0)$ cannot be predicted on the basis of its lagged values available at time *t*-1. So we try to confirm the independence of the score functions in time *t* and *t*-1.

The test statistic has a $\chi^2(l)$ asymptotic distribution in such a case and the following form

$$\left[T^{-1/2}\sum_{t=1}^{T}\mathbf{c_t}(\hat{\theta})\right]'\left[T^{-1}\sum_{t=1}^{T}\mathbf{c_t}(\hat{\theta})\cdot[\mathbf{c_t}(\hat{\theta})]'\right]^{-1}\left[T^{-1/2}\sum_{t=1}^{T}\mathbf{c_t}(\hat{\theta})\right]\rightarrow \chi^2(l). \qquad (9)$$

Hamilton in [4] used this testing for MSW models, where he derived autocorrelation tests inside and along regimes, ARCH effects test and the last one we are focusing on, the test for the validity of Markov assumptions. They have the form

$$P(s_t = j\,|\,s_{t-1} = i) = P(s_t = j\,|\,s_{t-1} = i, y_{t-1}) \qquad i, j = 1,..., N \qquad (10)$$

$$P(s_t = j\,|\,s_{t-1} = i) = P(s_t = j\,|\,s_{t-1} = i, s_{t-2} = k) \qquad i, j, k = 1,..., N. \qquad (11)$$

The first assumption means that transitional probabilities would not be dependent on the observable variable and the second one represents simply the first-order Markov property (1).

In the vector $\mathbf{c_t}(\hat{\theta})$ we must include the elements corresponding to the above mentioned examined assumptions

$$\frac{\partial \ln f(y_t\,|\,\Omega_{t-1};\hat{\theta})}{\partial p_{ij}}\cdot\frac{\partial \ln f(y_{t-1}\,|\,\Omega_{t-2};\hat{\theta})}{\partial \phi_{0,i}} \qquad i, j = 1,..., N, \qquad (12)$$

$$\frac{\partial \ln f(y_t\,|\,\Omega_{t-1};\hat{\theta})}{\partial p_{ij}}\cdot\frac{\partial \ln f(y_{t-1}\,|\,\Omega_{t-2};\hat{\theta})}{\partial p_{ij}} \qquad i, j = 1,..., N. \qquad (13)$$

The vector $\mathbf{c_t}(\hat{\theta})$ contains 2*N*(*N*-1) elements (after omitting redundant parameters), where *N* is the number of regimes, and then the test statistic has a $\chi^2(2N(N-1))$ asymptotic distribution. More details can be found in Hamilton [4].

4.3 Application

Firstly we apply this approach to testing linearity against Markov-switching type non-linearity and then we use the same idea for testing remaining non-linearity, where we compare a 2-regime model against a 3-regime model. We are trying to find out if two regimes are enough for description, or if it is necessary to add another one.

4.3.1 Testing Linearity against Markov-Switching Type Non-Linearity

A null hypothesis is represented by the validity of Markov assumptions in this case. If a 2-regime model does not confirm the examined assumptions (10) and (11), we reject the null hypothesis. This means that a linear model would be better because this time series does not show Markov-switching type non-linearity.

Firstly, we must calculate the test statistic (9) for the 2-regime model and find out its p-value from $\chi^2(4)$ distribution. If the p-value is less than the given significance level α, the null hypothesis will be rejected and we will conclude that for this time series it is better to use a linear model.

4.3.2 Testing Remaining Non-Linearity

We use the same principle for testing remaining non-linearity. So if the previous testing linearity does not reject the null hypothesis, we can continue and test for remaining non-linearity by comparing a 2-regime with a 3-regime model. We examine the appropriateness of a 2-regime model against an alternative hypothesis about a 3-regime model.

After calculating the test statistic (9) for the 3-regime model and corresponding p-values from $\chi^2(12)$ distribution (for the 2-regime model this is already calculated), we compare these *p*-values with the given significance level α. The following alternatives can occur:

- Non-rejecting the null hypothesis for a 2-regime model, rejecting null hypothesis for a 3-regime model. This means that the 2-regime model is appropriate.
- Non-rejecting the null hypothesis for a 2-regime model, non-rejecting the null hypothesis for a 3-regime model. The model with a greater *p*-value from testing the validity of Markov properties is the appropriate model. In this case we can check these results with other criterions such as the BIC (Bayesian Information Criterion) values for both types of models (the better model has a lower BIC value), residual dispersion, forecasting error values, results for testing autocorrelation, and so on.

5 Comparing the New Testing with Classical Simulation

To support our theory of new testing we calculate the classical test via simulation as is described in Section 3. We chose 100 time series from the financial and economic sectors.

The biggest advantage of the new testing procedure is the much shorter computation time. For instance see Table 1. We needed only 68.5*s* for the model order q = 5 (the length of the time series was 130) with the new approach testing linearity. On the other hand the simulation was computed in 14 802.7*s* for the same time series and the model order. For the alternative test of remaining non-linearity we needed 1 031.69*s* (the same time series) and the simulation experiment took 53 372 *s*. To understand better why the simulation takes so long, the reader is recommended to return back to Section 3, where the simulation experiment is described in detail.

Table 1

Comparing of computation time in both approaches

T=130, q=5	Computation time [s]	
	Testing linearity	Remaining non-linearity
Simulation	14 802.7	53 372
New approach	68.5	1 031.69

The main results are that the same conclusions are reached by both approaches. This means that an appropriateness of the linear model or non-linear model occurred in 72% of all cases. In the testing of remaining non-linearity we obtained the same conclusions in 79% of all cases. As an example of the test evaluation we present the results of the testing for the Russian rouble to Euro exchange rate time series in Table 2.

Table 2

Results of testing linearity and testing remaining non-linearity by both approaches for Rouble/EUR exchange rate

p-value for	Testing linearity		Testing remaining non-linearity	
Rouble	New approach	Simulation	New approach	Simulation
q=1	<0.001	0.981		
q=2	0.1657	<0.001	<0.001	0.97
q=3	0.1979	<0.001	0.065	0.048

We can see that for q=1, linearity was claimed for the examined time series by both approaches. So we do not test remaining non-linearity any further. For q=2 and q=3 the linear hypothesis was rejected. Then we continued with testing and the results were that for q=2, the Markov assumptions were not confirmed. The simulation also confirmed the sufficiency of the 2-regime model. In the case of q=3, we can see that the validity of the Markov assumptions was confirmed in both cases. We take the model with the higher p-value, but we should also support it with other criterions as we mentioned in the last part of Section 4.

Conclusions

Our main goal for proposing a new approach of testing linearity against Markov-switching type non-linearity was to reduce the computation time because it was very demanding to calculate it by simulation, particularly if more than one time series analysis was carried out.

Even though we cannot claim that both approaches are exactly the same tests or at least always substitutable, the new one could be very helpful in avoiding all simulations.

There are still some open problems. After these results we would like to follow up and calculate the power properties of both approaches and compare them with other types of non-linearity tests from [8]. Other interesting ideas for further research would be to investigate efficiency and to discover how the method works if the theoretical model is known. Next we should test the residuals and their independence, because correlation on its own is not sufficient. We could get linearly uncorrelated residuals but they can be dependent. Another issue was mentioned by Pál Rákonczai in 2008 [9], who suggested using auto-copulas instead of the autocorrelation function because the autocorrelation function describes only linear dependence and we need to describe non-linear dependence.

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