Combining Fuzzy/Wavelet Adaptive Error Tracking Control Design

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Abstract: A combining adaptive fuzzy-wavelet control algorithm is proposed for a class of continuous time unknown nonlinear systems. An application of wavelet networks to control problems of nonlinear systems is investigated in this work. A wavelet network is constructed as an alternative to a neural network to approximate a nonlinear system. Based on this wavelet network and fuzzy approximation, suitable adaptive control laws and appropriate parameter update algorithms for nonlinear uncertain (or unknown) systems are developed to achieve tracking performance. The stability analysis for the proposed control algorithm is provided. A nonlinear system simulation example is presented to verify the effectiveness of the proposed method.

Keywords: fuzzy control; adaptive control; wavelet approximation; feedback linearization

1 Introduction

In recent years, wavelet neural networks which combine the learning ability of feed forward neural networks and time-frequency localization properties of wavelets have become a popular tool for multiscale analysis and synthesis, time-frequency signal analysis in signal processing, function approximation, approximation in solving partial differential equations, and so on [1]-[8].

At present, there are two kinds of wavelet neural network structures. The first one is the fixed wavelet basis, where the dilation and translation parameters of wavelet basis are fixed, and the output layer weights are adjustable. The second one is the variable wavelet basis. The dilation parameters, translation parameters, and the output layer weights are adjustable in this type of wavelet neural network.

On the other hand, considerable study has been performed to integrate the excellent learning capability of neural networks with the perfect inference mechanism of fuzzy systems, which are called neuro-fuzzy systems [9], to obtain the rule-base membership function parameters from the input-output data. These neuro-fuzzy systems have fast and accurate learning and good generalization capabilities, and both have the ability to accommodate expert knowledge about the problem under consideration.

Fuzzy logic controllers are generally considered applicable to plants that are mathematically poorly understood and where experienced human operators are available. However, fuzzy controllers have not been regarded as an exact science due to the lack of a guarantee of global stability and acceptable performance. Nonetheless, some researchers propose the stability analysis of fuzzy control systems (e.g., [10]). The mathematical model of the plant is assumed to be known in [10]. Hence, this contradicts the very fundamental premise of fuzzy control systems. In fact, if the model of plant is known, then we should give the conventional linear or nonlinear control methods high priority.

The proposed control scheme provides good transient and robust performance. In this paper, it is proved that the closed-loop system is globally stable in the Lyapunov sense and the system output asymptotically stable with modeling uncertainties and disturbances.

Fuzzy controllers are assumed to work in situations where the plant parameters and structures have some uncertainties or unknown variations. The basic objective of adaptive control is to maintain the consistent performance of a system in the presence of uncertainties. So, advanced fuzzy control or wavelet approximation might be adaptive. This work is involved by combining the characteristics of wavelet, the technique of feedback linearizations, the adaptive control scheme and the fuzzy control to solve the tracking control design problem for nonlinear systems with bounded unknown or uncertain parameters and external disturbances.

This paper is organized as follows. First, the problem formulation is presented in Section 2. A brief description of a wavelet system is included in Section 3. In Section 4, the adaptive fuzzy-wavelet control is proposed. Simulation results for the proposed control concept are shown in Section 5. Finally, the paper is concluded in Section 6.

2 Problem Formulation

Consider an nth order SISO nonlinear system with $n \geq 2$ of the following form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_n &= f(\underline{x}) + g(\underline{x})u \\ y &= x_1 \end{aligned} \tag{1}$$

where $\underline{x} = [x, \dot{x}, \cdots, x^{(n-1)}]^T = [x_1, x_2, \cdots, x_n]^T \in R^n$ is the state vector, u is the control input and y is the output of the system. All the elements of the state vector $\underline{x}$ are assumed to be available. At the beginning, $f(\underline{x})$ is assumed to be smooth and $g(\underline{x})$ is assumed to be smooth and bounded away from zero. Differentiating the output y with respect to time for n times we obtain the following input/output form

$$y^{(n)} = f(\underline{x}) + g(\underline{x})u \tag{2}$$

Note that the above system has a relative degree of n.

If $f(\underline{x})$ and $g(\underline{x})$ are known, a nonlinear tracking control can be obtained. Let y_r be the desired continuous differentiable uniformly bounded trajectory and let

$$\underline{e} = y - y_r = (e, \dot{e}, \cdots, e^{(n-1)})^T \in R^n \tag{3}$$

be the tracking error. Then by employing the technique of feedback linearization a suitable control law can be derived to achieve the tracking control goal as

$$u = \frac{1}{g(\underline{x})}\left[-f(\underline{x}) + u_p + \nu\right] \tag{4}$$

where u_p is an auxiliary control variable yet to be specified and

$$\nu = y_r^{(n)} + \alpha_1\left(y_r^{(n-1)} - y^{(n-1)}\right) + \cdots + \alpha_n\left(y_r - y\right) \tag{5}$$

Note that the coefficients $\alpha_1, \ldots, \alpha_n$ are positive constants to be assigned such that the polynomial $s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$ is Hurwitz. As a result, the error dynamic of the system has the following input/output form

$$e^{(n)} + \alpha_1 e^{(n-1)} + \ldots + \alpha_n e = u_p \tag{6}$$

which can be represented in state space form as

$$\underline{\dot{e}} = \mathbf{A}\underline{e} + \mathbf{B}u_p \tag{7}$$

where

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix} \tag{8}$$

$$\mathbf{B}=\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}^T \tag{9}$$

$$\underline{e}=\begin{bmatrix}e & \cdots & e^{(n-2)} & e^{(n-1)}\end{bmatrix}^T \tag{10}$$

Note that the above design method is useful only if $f(\underline{x})$ and $g(\underline{x})$ are known exactly. If $f(\underline{x})$ and $g(\underline{x})$ are unknown then adaptive strategies must be employed. Let us now discuss a wavelet-network based adaptive algorithm.

First we employ two wavelet networks

$$\hat{f}\left(\underline{x},\underline{\theta}_f\right)=\underline{\theta}_f^T W_f\left(\underline{c}_f^T \underline{x}\right) \tag{11}$$

$$\hat{g}\left(\underline{x},\underline{\theta}_g\right)=\underline{\theta}_g^T W_g\left(\underline{c}_g^T \underline{x}\right) \tag{12}$$

to approximate (or model) the nonlinear functions $f(\underline{x})$ and $g(\underline{x})$ of the system, respectively.

3 A Review of Wavelet Networks

In this section a brief introduction to wavelet networks is given. Several kinds of wavelet bases have successfully been developed and widely applied in many different areas, such as in time-frequency signal analysis in signal processing, function approximation, approximation in solving partial differential equations and so on. Further development of new families of wavelet bases continues to receive considerable attention from researchers.

Consider the closed space U_i, $\forall i \in Z$ with the following properties [11]

$$U_i \quad \cdots \subset U_{-1} \subset U_0 \subset U_1 \cdots \tag{13}$$

$$\cap_{i \in Z} U_i = \{0\} \tag{14}$$

$$U_{i+1} = U_i \oplus W_i \quad \forall i \in Z \tag{15}$$

$$f(\underline{x}) \in U_i \Leftrightarrow f(2\underline{x}) \in U_{i+1} \quad \forall i \in Z \tag{16}$$

where Z is the set of all integers, $\cap$ is the intersection operator and $\oplus$ is the direct sum. It is seen that the decomposition of the whole space S can be rewritten as follows

$$S = U_i \oplus W_i \oplus W_{i+1} \oplus \cdots \oplus W_0 \oplus W_1 \oplus \cdots \tag{17}$$

for some $i \in Z$. Let $\phi(\underline{x}) \in S$ be a basic scaling function such that $U_i = \text{span}\{\phi_{ij}(\underline{x})\}$ with $\phi_{ij}(\underline{x}) = 2^{\frac{i}{2}}\phi(2^i\underline{x} - j)$, for all $i, j \in Z$; then, there exists a basic function $\psi(\underline{x}) \in S$ such that $W_i = \text{span}\{\psi_{ij}(\underline{x})\}$ with $\psi_{ij}(\underline{x}) = 2^{\frac{i}{2}}\psi(2^i\underline{x} - j)$, for all $i, j \in Z$.

Now, consider a function $f(\underline{x})$ is S. It is obvious that $f(\underline{x})$ can be rewritten as [11], [12]

$$f(\underline{x}) = \sum_i \sum_j \theta_{ij}\psi_{ij}(\underline{x}) \tag{18}$$

where

$$\theta_{ij} = \int_{-\infty}^{\infty} f(x)\psi_{ij}(x)dx \tag{19}$$

with $\psi_{ij}(\underline{x}) = 2^{\frac{i}{2}}\psi(2^i\underline{x} - j)$, for all $i, j \in Z$. The above expression of $f(\underline{x})$ is called a wavelet series expansion of the function $f(\underline{x})$.

Based on the wavelet series expansion, a wavelet network of the form [13], [14]

$$\hat{f}(\underline{x}, \underline{\theta}) = \sum_{i=M_1}^{M_2} \sum_{j=N_1}^{N_2} \theta_{ij}\psi_{ij}(\underline{x}) = \underline{\theta}^T W(\underline{x}) \tag{20}$$

can be constructed to approximate a nonlinear function $f(\underline{x})$ in space S, for some integers M_1, M_2, N_1 and N_2 where

$$\underline{\theta} = \left[\theta_{M_1N_1} \cdots \theta_{M_1N_2} \cdots \theta_{M_2N_1} \cdots \theta_{M_2N_2}\right]^T \tag{21}$$

and

$$W(x) = \left[\psi_{M_1N_1}(x) \ldots \psi_{M_1N_2}(x) \ldots \psi_{M_2N_1}(x) \ldots \psi_{M_2N_2}(x)\right]^T \tag{22}$$

This wavelet network represents an alternative to a neural network approximation. If $\epsilon(M_1, M_2, N_1, N_2) = f(x) - \hat{f}(\underline{x}, \underline{\theta})$ is the approximation error, then for arbitrary constant $\varepsilon \geq 0$ there exist some constants $M_1, M_2, N_1, N_2 \in Z$ such that $\left\|\epsilon(M_1, M_2, N_1, N_2)\right\|_2 \leq \varepsilon$, for all c in compact set $X \subset R$. This means that the wavelet network $\hat{f}(\underline{x}, \underline{\theta})$ can approximate $f(\underline{x})$ to any desired accuracy.

In the case of a function $f(\underline{x})$ defined on $X \subset R^n$ with $\underline{x} = [x_1, x_2, \cdots, x_n]^T$, the proposed wavelet network $\hat{f}(\underline{x}, \underline{\theta})$ cannot be applied directly because $\hat{f}(\underline{x}, \underline{\theta})$ is defined on $X \subset R$, not on $X \subset R^n$. We must first make a minor modification by replacing the wavelet bases in Eq. (20) by $\psi_{ij}\left(\underline{c}^T, \underline{x}\right) = \psi_{ij}\left(\sum_{i=1}^{n} c_i x_i\right)$ with some weighting constants c_i.

Then the modified wavelet network becomes

$$\hat{f}(\underline{x}, \underline{\theta}) = \sum_{i=M_1}^{M_2} \sum_{j=N_1}^{N_2} \theta_{ij}\psi_{ij}\left(\underline{c}^T \underline{x}\right) = \underline{\theta}^T W\left(\underline{c}^T \underline{x}\right) \tag{23}$$

Note that this modified wavelet network is composed of four layers. The first layer is the input layer with available input vector $\underline{x} = [x_1, x_2, \cdots, x_n]^T$. A weighting summer $\underline{c}^T \underline{x}$ is given in the second layer. The third layer is composed of the wavelet bases. The output layer is a weighted combination of the wavelets.

4 Adaptive Fuzzy/Wavelet Control

According to the description in Section 3, guaranteeing $\underline{x}$ in a compact region is very important when the wavelet networks $\hat{f}\left(\underline{x},\underline{\theta}_f\right)$ and $\hat{g}\left(\underline{x},\underline{\theta}_g\right)$ are used to approximate $f\left(\underline{x}\right)$ and $g\left(\underline{x}\right)$, respectively. In general there is still not an efficient way to ensure satisfaction of this requirement. In practical applications one may assign a very large compact set to avoid violation of this requirement. However, a very large wavelet basis is needed in this situation. This may result in a large computational burden. Fortunately, in many physical systems such as mechanical systems and electrical systems, an appropriate selection of the pre-assigned compact set can be obtained via knowledge of some physical limitations.

Let

$$\underline{\theta}_f^* = \arg\min_{\underline{\theta}_f}\max_{\underline{x}}\left|\hat{f}\left(\underline{x},\underline{\theta}_f\right)-f\left(\underline{x}\right)\right| \tag{24}$$

$$\underline{\theta}_g^* = \arg\min_{\underline{\theta}_g}\max_{\underline{x}}\left|\hat{g}\left(\underline{x},\underline{\theta}_g\right)-g\left(\underline{x}\right)\right| \tag{25}$$

be the best approximation parameters of $\underline{\theta}_f$ and $\underline{\theta}_g$, respectively.

System (1) can be rewritten as

$$x_1^{(n)} = f\left(x_1,\ldots,x_n\right)+g\left(x_1,\ldots,x_n\right)u \tag{26}$$

where $\underline{x}=[x,\dot{x},\cdots,x^{(n-1)}]^T=[x_1,x_2,\cdots,x_n]^T\in R^n$ is the state vector and the functions $f\left(\underline{x}\right)$ and $g\left(\underline{x}\right)$ are unknown nonlinear functions of the states and time. The objective of the adaptive wavelet error tracking control design is to update the controller parameters in such as a way that the system output can asymptotically track the desired reference model output $y_r = x_m\left(t\right)$ in spite of function uncertainties.

The reference model is a linear system in form

$$x_{m_1}^{(n)}+a_{n-1}x_{m_1}^{(n-1)}+\cdots+a_1\dot{x}_{m_1}+a_0x_{m_1}=br \tag{27}$$

where $\underline{x}_m=[x_{m_1},\dot{x}_{m_1},\cdots,x_{m_1}^{(n-1)}]^T=[x_{m_1},x_{m_2},\cdots,x_{m_n}]^T\in R^n$ is the state vector of the reference model.

To follow the reference model, the controller must be chosen so as to cancel the nonlinearities in the nonlinear system and provide pole placement to the system, i.e. feedback linearization. For example, the controller is chosen in the form

$$u = \frac{1}{\hat{g}(\underline{x})}\left[-\hat{f}(\underline{x}) - \hat{a}_{n-1}x_1^{(n-1)} - \cdots - \hat{a}_1\dot{x}_1 - \hat{a}_0 x_1 + \hat{b}r\right] \tag{28}$$

In this article the set of fuzzy systems is used with a singleton fuzzifier, product inference, a centroid defuzzifier, a triangular antecendent membership function and a singleton consequent membership function with n inputs of $x_i \in \left[c_{x_i} - k_{x_i}, c_{x_i} + k_{x_i}\right]$ for $i = 1,\ldots,n$ and $\bar{u} \in [0,1]$ as the normalized output. The generalized expression of the class of the fuzzy controllers can be written as

$$\bar{u} = \sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2} N_{i_1\cdots i_n} x_1^{i_1-1}\cdots x_n^{i_n-1} \tag{29}$$

$$N_{i_1\cdots i_n} = \frac{\left[\sum_{j_1=1}^{2}\cdots\sum_{j_n=1}^{2} R_{j_1\cdots j_n} K_{j_1\cdots j_n} C_{j_1\cdots j_n}\right]}{2^n \prod_{i=1}^{n} k_{x_i}} \tag{30}$$

$$C_{j_1\cdots j_n} = \left[\frac{(-1)^{j_1}}{k_{x_1} - (-1)^{j_1} c_{x_1}}\right]^{i_1-1} \cdots \left[\frac{(-1)^{j_n}}{k_{x_n} - (-1)^{j_n} c_{x_n}}\right]^{i_1-1} \tag{31}$$

$$K_{j_1\cdots j_n} = \left[k_{x_1} - (-1)^{j_1} c_{x_1}\right]\cdots\left[k_{x_n} - (-1)^{j_n} c_{x_n}\right] \tag{32}$$

On the other hand, given the coefficients of the explicit form $N_{i_1\cdots i_n}$ we can reconstruct the rule base from the generalized expression of the class of fuzzy systems [15] by using the following theorem.

Theorem 1 [15]: For a class of fuzzy logic systems (FLS) with a singleton fuzzifier, product inference, a centroid defuzzifier, a triangular antecendent membership function and a singleton consequent membership function, i.e. given the coefficients of the explicit form, i.e. $N_{i_1\cdots i_n}$, the control function can be expressed in terms of fuzzy rules as

$$R_{j_1\cdots j_n} = \sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2} N_{i_1\cdots i_n} D_{j_1\cdots j_n} \tag{33}$$

with

$$D_{j_1\cdots j_n} = \left[c_{x_1} + (-1)^{j_1} k_{x_1}\right]^{i_1-1}\cdots\left[c_{x_n} + (-1)^{j_n} k_{x_n}\right]^{i_n-1} \tag{34}$$

Proof: The proof is found by directly expanding terms and comparing coefficients. For details, please refer to [15].

Therefore, one can express an equation in the form of generalized multilinear equations, such as polynomials, exactly as a rule base of FLS. Theorem 1 is useful in cases where the implementation of an FLS performs inference on a given fuzzy rule base but without any numerical computation capability.

We can express the fuzzy controller in the form of fuzzy IF-THEN rules.

RULE i: IF r is A_1^r and ... and x_n is $A_1^{x_n}$, THEN $\overline{u}_p = R_i$

The generalized expression of the class of fuzzy controller with n+1 inputs, i.e. r and $\underline{x}$ can be written as

$$\overline{u}_p = \sum_{i_0=1}^{2} \cdots \sum_{i_n=1}^{2} N_{i_0 i_1 \cdots i_n} r^{i_0-1} x_1^{i_1-1} \cdots x_n^{i_n-1} \tag{35}$$

By applying Theorem 1, one can find a set of R_i's to represent exactly the given pole-placement equation as $u_p = -\widehat{a}_{n-1} x_1^{(n-1)} - \cdots - \widehat{a}_1 \dot{x}_1 - \widehat{a}_0 x_1 + \widehat{b} r$.

The controller for pole-placement can be written as

$$u_p = \underline{\theta}_p^T \underline{\omega}_p \tag{36}$$

with $\underline{\theta}_p^T = \left(k_0, \underline{k}_b^T, \underline{k}_c^T\right)$

and $\underline{\omega}_p^T = \left(r, \underline{x}^T, \underline{x}_c^T\right)$

with

$$k_0 = 2N_{211\ldots111}$$

$$k_1 = 2N_{121\ldots111}$$

$$k_{n-1} = 2N_{111\ldots121}$$

$$k_n = 2N_{111\ldots112}$$

where $\underline{k}_b = \left[k_1, \cdots, k_n\right]^T$. The composite state vector $\underline{x}_c$ and the associated parameter vector $\underline{k}_c$ are defined as

$$\underline{x}_c^T = \left(r x_1 x_2 \ldots x_n, r x_1 x_2 \ldots x_{n-1}, \cdots, x_{n-1} x_n, 1\right) \tag{37}$$

$$\underline{k}_c^T = \left(k_{n+1}, k_{n+2}, \cdots, k_{n+n_c-1}, k_{n+n_c}\right) \tag{38}$$

with

$$k_{n+1} = 2N_{222\ldots222}$$

$$k_{n+2} = 2N_{222\ldots221}$$

$$k_{n+n_c-1} = 2N_{111\ldots122}$$

$$k_{n+n_c} = 2N_{111\ldots111}$$

where $n_c = 2^{n+1} - (n+1)$

Controller can be stated as

$$u = \frac{1}{\hat{g}(\underline{x})}\left[u_p - \hat{f}(\underline{x})\right] \tag{39}$$

From the nonlinear system (26) we have

$$\begin{aligned} x_1^{(n)} &= f(\underline{x}) + g(\underline{x})u \\ &= f(\underline{x}) + g(\underline{x})u - \hat{g}(\underline{x})u + \hat{g}(\underline{x})u \\ &= f(\underline{x}) + \hat{g}(\underline{x})u + \left(g(\underline{x}) - \hat{g}(\underline{x})\right)u \end{aligned} \tag{40}$$

By substituting (39) into the previous equation it becomes

$$x_1^{(n)} = \underline{k}_b^T \underline{x} + k_0 r + \underline{k}_c^T + \left(f(\underline{x}) - \hat{f}(\underline{x})\right) + \left(g(\underline{x}) - \hat{g}(\underline{x})\right)u \tag{41}$$

By substracting the closed-loop plant dynamic equation (above) with the reference model dynamic (27) we have the following

$$
\begin{aligned}
x_1^{(n)} - x_{m_1}^{(n)} &= \underline{k}_b^T \underline{x} + k_0 r + \underline{k}_c^T \underline{x}_c \\
&+ \left(f(\underline{x}) - \hat{f}(\underline{x})\right) + \left(g(\underline{x}) - \hat{g}(\underline{x})\right) u \\
&+ \sum_{j=1}^{n-1} a_j x_{m_1}^{(j)} - br \\
&= -\sum_{j=1}^{n-1} \left[a_j \left(x_1^{(j)} - x_{m_1}^{(j)} \right) \right] \\
&+ \sum_{j=1}^{n-1} \left[\left(k_j + a_j \right) x_1^{(j)} \right] \\
&+ \left(k_0 - b\right) r + \underline{k}_c^T \underline{x}_c \\
&+ \left(f(\underline{x}) - \hat{f}(\underline{x})\right) + \left(g(\underline{x}) - \hat{g}(\underline{x})\right) u
\end{aligned}
\tag{42}
$$

For the time derivative of the signal error vector $\underline{e} = \underline{x} - \underline{x}_m$ the following equality holds

$$
\begin{aligned}
e_1^{(n)} = &-\sum_{j=1}^{n-1} \left[a_j e_1^{(j)} \right] + \sum_{j=1}^{n-1} \left[\left(k_j + a_j \right) x_1^{(j)} \right] \\
&+ \left(k_0 - b\right) r + \underline{k}_c^T \underline{x}_c + \left(f(\underline{x}) - \hat{f}(\underline{x})\right) \\
&+ \left(g(\underline{x}) - \hat{g}(\underline{x})\right) u
\end{aligned}
\tag{43}
$$

We can rewrite the error (43) in matrix representation

$$
\dot{\underline{e}} = \mathbf{A}_m \underline{e} + \underline{b}_I \underline{\phi}^T \underline{\omega}
\tag{44}
$$

The error vector $\underline{e}$ is defined as

$$
\underline{e} = \begin{pmatrix} e_1 \\ \dot{e}_1 \\ \vdots \\ e_1^{(n-1)} \end{pmatrix} = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ \vdots \\ x_1^{(n-1)} \end{pmatrix} - \begin{pmatrix} x_{m_1} \\ \dot{x}_{m_1} \\ \vdots \\ x_{m_1}^{(n-1)} \end{pmatrix}
\tag{45}
$$

The matrix $\mathbf{A}_m$ and vector $\underline{b}_I$ are defined as

$$\mathbf{A}_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{pmatrix} \tag{46}$$

$$\underline{b}_I = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{47}$$

with the parameter error vector $\underline{\phi}$ defined as

$$\underline{\phi}^T = \begin{Bmatrix} k_0 - b & k_1 + a_1 & \cdots & k_n + a_n \\ k_{n+1} & k_{n+2} & \cdots & k_{n+n_c} \\ \left(\underline{\theta}_f^* - \underline{\theta}_f\right)^T & \left(\underline{\theta}_g^* - \underline{\theta}_g\right)^T & & \end{Bmatrix} \tag{48}$$

$$\underline{\omega}^T = \begin{Bmatrix} r & x_1 & \cdots & x_n \\ rx_1x_2\ldots x_n & rx_1x_2\ldots x_{n-1} & \cdots & 1 \\ W_f\left(\underline{c}_f^T\underline{x}\right) & W_g\left(\underline{c}_g^T\underline{x}\right) & & \end{Bmatrix} \tag{49}$$

where $\underline{\theta}_f^* W_f\left(\underline{c}_f^T\underline{x}\right) \approx f\left(\underline{x}\right)$ and $\underline{\theta}_g^* W_g\left(\underline{c}_g^T\underline{x}\right) \approx g\left(\underline{x}\right)$. The system's error (44) consists of a linear part governed by $\mathbf{A}_m$ and $\underline{b}_I$ plus a nonlinear control $\underline{\phi}^T\underline{\omega}$. In the following we show stable adaptive laws for the system.

Theorem 2: Consider the error equation given by (43) whose parameters are adjusted according to the following adaptive laws.

1) For the nonlinear-cancellation for $f\left(\underline{x}\right)$ the adaptive law is

$$\dot{\underline{\theta}}_f = -\gamma\left(\underline{p}^T\underline{e}\right)W_f\left(\underline{c}_f^T\underline{x}\right)$$

2) For the nonlinear-cancellation for $g\left(\underline{x}\right)$ the adaptive law is

$$\dot{\underline{\theta}}_g = -\gamma\left(\underline{p}^T\underline{e}\right)W_g\left(\underline{c}_g^T\underline{x}\right)$$

Then we have

1) $\underline{e}$ and $\underline{\phi}$ are uniformly bounded

2) $\lim_{t\to\infty} \underline{e} = 0$

where $\underline{p}$ is a vector consisting of the n-th column of positive definite symmetric matrix **P** (see Eq. 45).

Proof: The choice of the Lyapunov function is normally a quadratic function of both the signal error vector $\underline{e}$ and the parameter error $\underline{\phi}$

$$V = \underline{e}^T \mathbf{P} \underline{e} + \underline{\phi}^T \mathbf{\Gamma}^{-1} \underline{\phi} \tag{50}$$

with the adaptation gain matrix defined as $\mathbf{\Gamma} = \gamma \mathbf{I}_{2^{n+1}\times 2^{n+1}}$, where $\mathbf{I}_{2^{n+1}\times 2^{n+1}}$ is a $2^{n+1} \times 2^{n+1}$ identity matrix. Since $\mathbf{\Gamma}$ is positive definite, $\mathbf{\Gamma}^{-1}$ is also positive definite. Matrix **P** must be chosen as a positive definite symmetric matrix and it will follow from the adaptive law derivation shown in the following. To obtain an asymptotically stable adaptive system, $\dot{V}$ must be negative definite. Differentiating V yields with

$$\dot{V} = \underline{e}^T \left(\mathbf{A}_m^T \mathbf{P} + \mathbf{P} \mathbf{A}_m \right) \underline{e} + 2\underline{e}^T \mathbf{P} \underline{b}_I \underline{\phi}^T \underline{\omega} + 2\underline{\phi}^T \mathbf{\Gamma}^{-1} \dot{\underline{\phi}} \tag{51}$$

By applying the second method of Lyapunov, positive definite symmetric matrices **P** and **Q** can be found such that the first part of the equation satisfies

$$\underline{e}^T \left(\mathbf{A}_m^T \mathbf{P} + \mathbf{P} \mathbf{A}_m \right) \underline{e} = -\underline{e}^T \mathbf{Q} \underline{e} \tag{52}$$

By putting the last two terms of the equation to zero the adaptive laws emerges

$$\begin{aligned} 2\underline{e}^T \mathbf{P} \underline{b}_I \underline{\phi}^T \underline{\omega} + 2\underline{\phi}^T \mathbf{\Gamma}^{-1} \dot{\underline{\phi}} &= 0 \\ \dot{\underline{\phi}} &= -\mathbf{\Gamma} \underline{e}^T \mathbf{P} \underline{b}_I \underline{\omega} \\ &= -\mathbf{\Gamma} \left(\underline{p}^T \underline{e} \right) \underline{\omega} \end{aligned} \tag{53}$$

The product $\mathbf{P}\underline{b}_I$ is a vector consisting of the n-th column p of **P**, while the model and process parameters are assumed constant. From the definition of $\underline{\phi}$, it follows that

$$\dot{\underline{\theta}} = \mathbf{\Gamma}' \left(\underline{p}^T \underline{e} \right) \underline{\omega} \tag{54}$$

with $\boldsymbol{\Gamma}' = \dfrac{\boldsymbol{\Gamma}}{b_{pn}}$. By partitioning the parameter vectors, we can obtain the adaptive laws for the parameters of the two approximators. Since $\dot{V} < 0$ from (51) we obtain that $\underline{e}$ and $\underline{\phi}$ are uniformly bounded. Because of the boudedness of $\underline{e}$, $\underline{\phi}$ and $\underline{\omega}$ we see from (43) that $\dot{\underline{e}}$ is bounded as well. Thus $\underline{e}$ is uniformly continuous and so is $\dot{V}\left(\underline{e}, \underline{\phi}\right)$. From the fact that

$$V = \underline{e}^T \mathbf{P} \underline{e} + \underline{\phi}^T \boldsymbol{\Gamma}^{-1} \underline{\phi} \tag{55}$$

$$\dot{V} = -\underline{e}^T \mathbf{Q} \underline{e} \tag{56}$$

we have that

$$\lim_{t \to \infty} V = V^* \tag{57}$$

exists, with

$$V^* - V_0 = -\int_0^\infty \underline{e}^T \mathbf{Q} \underline{e} dt \tag{58}$$

Since the left-hand side is known to be finite, we know that the term on the right-hand side must be finite. We known that since $\underline{e}^T \mathbf{Q} \underline{e}$ is positive, uniformly continuous and has a finite integral that

$$\lim_{t \to \infty} \underline{e}^T \mathbf{Q} \underline{e} = 0 \tag{59}$$

and thus

$$\lim_{t \to \infty} \underline{e} = 0 \tag{60}$$

Notice that the sign of the actual adaptation gain matrix $\boldsymbol{\Gamma}'$ is found to depend on the sign of b_{pn} and so to be able to implement the adaptive law with a proper sign, the sign of b_{pn} must be known. This condition appears in all MRAC schemes. The equations form the adaptive laws that provide a stable adaptive system. The matrix **P** and so the vector $\underline{p}$ can be calculated with Lyapunov's equation starting with a chosen definite symetric matrix **Q**. Furthermore, the product of vectors $\left(\mathbf{P}\underline{b}_I\right)^T \underline{e}$ is called the "compensated error" in adaptive control literature. This adaptive law has the same form as the MIT adaptive laws, which use the error $\underline{e}$ instead of the compensated error $\underline{p}^T \underline{e}$. Since it can be shown that using the compensated error in the adaptation laws preserves the system stability, the word "compensated" refers to the compensation of the error in order to preserve system stability.

5 Simulation Example

Example 1

The above described adaptive fuzzy/wavelet control algorithm will now be evaluated using the inverted pendulum system depicted in Fig. 1.

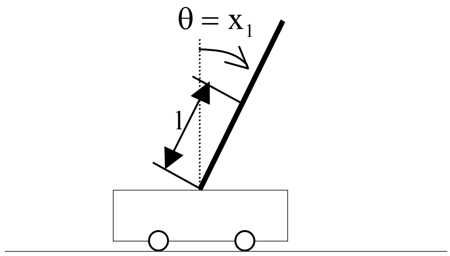

Figure 1
The inverted pendulum system

Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. The dynamic equation of the inverted pendulum is given by [16]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g \sin x_1 - \dfrac{m l x_2^2 \cos(x_1)\sin(x_1)}{m_c + m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2(x_1)}{m_c + m}\right)} \\ &\quad + \frac{\dfrac{\cos(x_1)}{m_c + m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2(x_1)}{m_c + m}\right)} u_c + d \\ y &= x_1 \end{aligned} \tag{61}$$

where g is the acceleration due to gravity, m_c denotes the mass of the cart, m is the mass of the pole, l is the half-length of the pole, the force u_c represents the control signal and d is the external disturbance. In simulations the following parameter values are used: $m_c = 1Kg$, $m = 0.1Kg$ and $l = 0.5m$. The reference signal is assumed to be $y_r(t) = (\pi/30)\sin(t)$ and an external disturbance $d(t) = 0.1\sin(t)$.

If we require

$$|\underline{x}| \leq \frac{\pi}{6},\ |u| \leq 180 \tag{62}$$

and substitute the functions sin(.) and cos(.) by their bounds, we can determine the bounds

$$f^M(x_1, x_2) = 15.78 + 0.366x_2^2 \tag{63}$$

$$g^M(x_1, x_2) = 1.46,\ g_m(x_1, x_2) = 1.12 \tag{64}$$

$k_1 = 2$, $k_2 = 1$ and $Q = diag(10,10)$ are set. Then the algebraic Riccati equation solution is $P = \begin{bmatrix} 15 & 5 \\ 5 & 5 \end{bmatrix}$ and $\lambda_{min}(P) = 2.93$. To satisfy the constraint related to $|\underline{x}|$ we choose $M_f = 16$, $M_g = 1.6$ and $\gamma = 0.48$. Five Gaussian membership functions for both x_1 and x_2 (i=1,2) are selected to cover the whole universe of discourse

$$\mu_{F_i^1}(x_i) = \exp\left(-\left(\frac{x_i - \pi/6}{\pi/24}\right)^2\right) \tag{65}$$

$$\mu_{F_i^2}(x_i) = \exp\left(-\left(\frac{x_i - \pi/12}{\pi/24}\right)^2\right) \tag{66}$$

$$\mu_{F_i^3}(x_i) = \exp\left(-\left(\frac{x_i}{\pi/24}\right)^2\right) \tag{67}$$

$$\mu_{F_i^4}(x_i) = \exp\left(-\left(\frac{x_i + \pi/12}{\pi/24}\right)^2\right) \tag{68}$$

$$\mu_{F_i^5}(x_i) = \exp\left(-\left(\frac{x_i + \pi/6}{\pi/24}\right)^2\right) \tag{69}$$

Using the method of trial and error $\gamma_f = 50$ and $\gamma_g = 1$ are chosen. The pendulum initial position is chosen as far as possible $(\theta(0) = x_1 = \pi/20)$ to emphasize the efficiency of our algorithm.

The Haar wavelets are chosen to be the basis of the wavelet network. The vectors $\underline{c}_f$ and $\underline{c}_g$ are both chosen as $\underline{c}_f = \underline{c}_g = \underline{c} = [1 \quad 1]^T$, and the size of our network is chosen as $M_1 = -2$, $M_2 = 2$, $N_1 = -1$ and $N_2 = 1$. In this example, the wavelet bases for $f(\underline{x})$ and $g(\underline{x})$ are chosen and are the same. Therefore, $W_f\left(\underline{c}_f^T \underline{x}\right) = W_g\left(\underline{c}_g^T \underline{x}\right) = W\left(\underline{c}^T \underline{x}\right)$.

Two cases have been considered in order to show the influence of the linguistic rules incorporation into the control law:

Case one: the initial values of $\underline{\theta}_f$ and $\underline{\theta}_g$ are chosen arbitrarily.

Case two: the initial values of $\underline{\theta}_f$ and $\underline{\theta}_g$ are deduced from the fuzzy rules describing the system dynamic behavior. For example, if we consider the unforced system, i.e. $u_c = 0$, the acceleration is equal to $f(x_1, x_2)$. Thus we can state intuitively:

"The bigger is x_1, the larger is $f(x_1, x_2)$".

Transforming this fuzzy information into a fuzzy rule we obtain

$R_f^{(1)}$: IF x_1 is F_1^5 and x_2 is F_2^5, THEN $f(x_1, x_2)$ is Positive Big

where "Positive Big" is a fuzzy set whose membership function is $\mu_{F_i^l}(x_i)$ given by (65)-(69). The acceleration is proportional to the gravity, i.e. $f(x_1, x_2) \cong \alpha \sin(x_1)$, where α is a constant. As $f(x_1, x_2)$ achieves its maximum at $x_1 = \pi/2$, using (63)-(64) we obtain $\alpha \cong 16$. The resulting set of 25 fuzzy rules characterizing $f(x_1, x_2)$ is given in Tab. 1.

Table 1
Linguistic rules for $f(x_1, x_2)$

$f(x_1, x_2)$			x_1				
			F_1^1	F_1^2	F_1^3	F_1^4	F_1^5
			$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$
x_2	F_2^1	$-\frac{\pi}{6}$	-8	-4	0	4	8
	F_2^2	$-\frac{\pi}{12}$	-8	-4	0	4	8
	F_2^3	0	-8	-4	0	4	8
	F_2^4	$\frac{\pi}{12}$	-8	-4	0	4	8
	F_2^5	$\frac{\pi}{6}$	-8	-4	0	4	8

Now the following observation is used to determine the fuzzy rules for $g(x_1, x_2)$:

"The smaller is x_1, the larger is $g(x_1, x_2)$".

Similarly to the case of $f(x_1, x_2)$ and based on the bounds (63)-(64) this observation can be quantified into the 25 fuzzy rules summarized in Tab. 2.

Table 2
Linguistic rules for $g(x_1, x_2)$

$g(x_1, x_2)$			x_1				
			F_1^1	F_1^2	F_1^3	F_1^4	F_1^5
			$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$
x_2	F_2^1	$-\frac{\pi}{6}$	1.26	1.36	1.46	1.36	1.26
	F_2^2	$-\frac{\pi}{12}$	1.26	1.36	1.46	1.36	1.26
	F_2^3	0	1.26	1.36	1.46	1.36	1.26
	F_2^4	$\frac{\pi}{12}$	1.26	1.36	1.46	1.36	1.26
	F_2^5	$\frac{\pi}{6}$	1.26	1.36	1.46	1.36	1.26

To obtain the same tracking performances the attenuation level ρ is equal to 0.2 in the first case and to 0.8 in the second one.

The tracking performance of both cases for a sinusoidal trajectory is illustrated in Fig. 2.

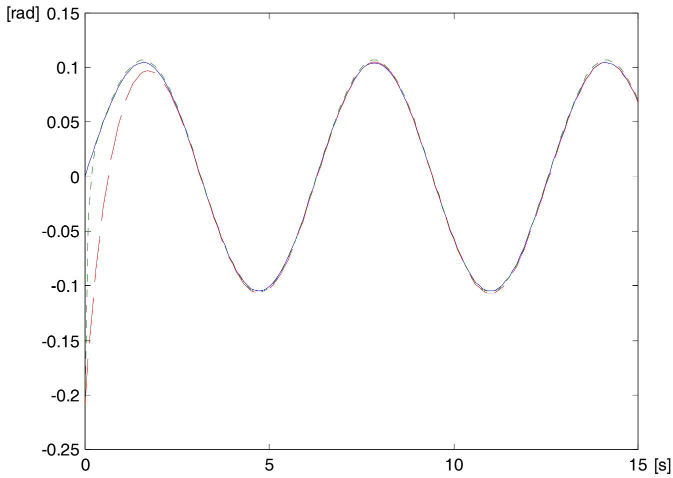

Figure 2

The state x_1 in case 1(red dashed line), in case 2 (green dotted line) and desired value $y_r(t)$ (blue solid line) for $\underline{x}(\underline{0}) = (\pi/12,0)^T$

Example 2

In this example, we apply the adaptive fuzzy/wavelet controller to the system

$$y'' + \frac{1}{0.25 + y} y' + 1.7y - 0.5u = 0 \tag{70}$$

Define six fuzzy sets over interval <-10, 10> with labels N3, N2, N1, P1, P2, P3. The membership functions are

$$\mu_{N1}(x) = \frac{1}{e^{(x+0.5)^2}} \tag{71}$$

$$\mu_{N2}(x) = \frac{1}{e^{(x+1.5)^2}} \tag{72}$$

$$\mu_{N3}(x) = \frac{1}{1+e^{5(x+2)}} \tag{73}$$

$$\mu_{P1}(x) = \frac{1}{e^{(x-0.5)^2}} \tag{74}$$

$$\mu_{P2}(x) = \frac{1}{e^{(x-1.5)^2}} \tag{75}$$

$$\mu_{P3}(x) = \frac{1}{1+e^{-5(x-2)}} \tag{76}$$

The reference model is assumed to be

$$M(s) = \frac{1}{s^2 + 2s + 1} \tag{77}$$

and the reference signal is the square periodic signal of magnitude 1.5 and frequency 0.01 Hz.

We choose $P = \begin{bmatrix} 50 & 30 \\ 30 & 20 \end{bmatrix}$, $k_1 = 2$, $k_2 = 1$, and $\lambda_{min}(P) = 1.52$. To satisfy the constraint related to $|\underline{x}|$ we choose $\overline{V} = 0.25$, $M_f = 20$, $M_g = 2.1$ and $\gamma = 0.25$.

At the 200th second of simulation the system (64) was switched to another system

$$y''' + 5y'' + \left[\frac{1}{(0.25+y)^2} - 1.7\right] y' + y - 5u = 0 \tag{78}$$

All initial states have been set to zero $y(0) = y'(0) = y''(0) = y'''(0) = 0$.

As can be seen from Fig. 3, the simulation results confirm the good adaptation capability of the proposed control system. The system dynamic changes are in particular manifested by changes of the control input signal (Fig. 4).

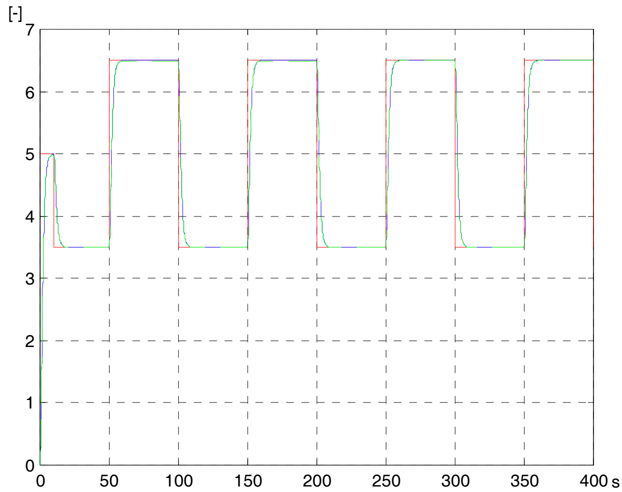

Figure 3
The state x_1 (blue dashed line), its desired reference model value $y_m(t)$ (green solid line) and reference signal (red solid line)

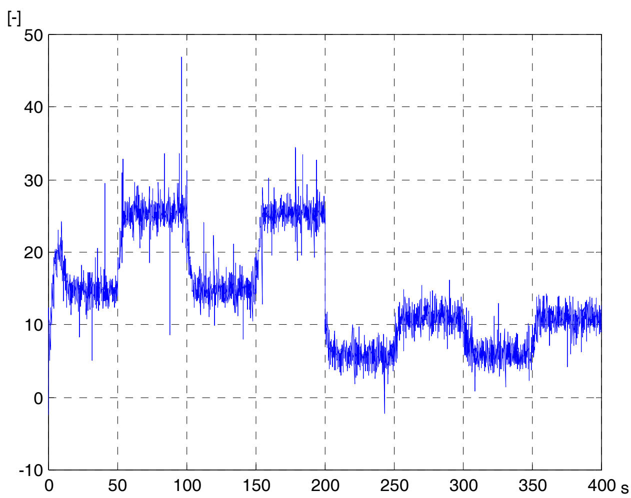

Figure 4
Control signal

Conclusions

The adaptive control technique has been combined with a wavelet network algorithm and a fuzzy approximation method in this study to achieve the desired attenuation of disturbance due to the approximation error and external noise in a class of nonlinear system under a large uncertainty or unknown variation in plant parameter and structure. The major advantage lies in that the accurate mathematical model of the system is not required to be known. The proposed method can guarantee the global stability of the resulting closed-loop system in the sense that all signals involved are uniformly bounded. In addition, the specific formula for the bounds is also given. Finally, the indirect adaptive controller has been used to control a nonlinear system to the origin.

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