Graphs with Equal Irregularity Indices

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Abstract: The *irregularity* of a graph can be defined by different so-called *graph topological indices*. In this paper, we consider the irregularities of graphs with respect to the *Collatz-Sinogowitz index* [8], the *variance of the vertex degrees* [6], the *irregularity of a graph* [4], and the *total irregularity of a graph* [1]. It is known that these irregularity measures are not always compatible. Here, we investigate the problem of determining pairs or classes of graphs for which two or more of the above mentioned irregularity measures are equal. While in [17] this problem was tackled in the case of bidegreed graphs, here we go a step further by considering tridegreed graphs and graphs with arbitrarily large degree sets. In addition we pressent the smallest graphs for which all above irregularity indices are equal.

Keywords: irregularity measures of graph; topological graph indices

1 Introduction

Let G be a simple undirected graph of order $n = |V(G)|$ and size $m = |E(G)|$. The *degree* of a vertex v in G is the number of edges incident with v and it is denoted by $d_G(v)$. A graph G is *regular* if all its vertices have the same degree, otherwise it is *irregular*. However, in many applications and problems it is of big importance to know how irregular a given graph is. The quantitative topological characterization of irregularity of graphs has a growing importance for analyzing the structure of deterministic and random networks and systems occurring in chemistry, biology and social networks [7, 12]. In this paper, we consider four graph topological indices that quantify the irregularity of a graph. Before we introduce those indices, we present some necessarily notions and definitions.

A *universal* vertex is the vertex adjacent to all other vertices. We denote by $m_{r,s}$ the number of edges in G with end-vertex degrees r and s, and by n_r the numbers of vertices n G with degree r. Numbers $m_{r,s}$ and n_r are referred as the *edge-parameters* and the *vertex-parameters* of G, respectively.

The *mean degree* of a graph G is defined as $\overline{d}(G) = 2m/n$. Graphs G_1 and G_2 are said to be *edge-equivalent* if for their corresponding edge-parameters sets $\{m_{r,s}(G_1) > 0\} = \{m_{r,s}(G_2) > 0\}$ holds. Analogously, they are called *vertex-equivalent* if for their vertex-parameters sets $\{n_r(G_1) > 0\} = \{n_r(G_2) > 0\}$ is fulfilled.
A sequence of non-negative integers $D = (d_1, d_2, \ldots, d_n)$ is said to be *graphical* if there is a graph with n vertices such that vertex i has degree d_i. If in addition $d_1 \geq d_2 \geq \cdots \geq d_n$ then D is a *degree sequence*. The *degree set*, denoted by $\mathcal{D}(G)$, of a simple graph G is the set consisting of the distinct degrees of vertices in G.
The *adjacency matrix* $A(G)$ of a simple undirected graph G is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position (v_i, v_j) according to whether v_i and v_j are adjacent or not. The *characteristic polynomial* $\phi(G, t)$ of G is defined as characteristic polynomial of $A(G)$: $\phi(G, \lambda) = \det(\lambda \mathbf{I_n} - A(G))$, where $\mathbf{I_n}$ is $n \times n$ identity matrix. The set of eigenvalues of the adjacent matrix $A(G)$ is called the *graph spectrum* of G. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called the *spectral radius* of G. Graphs that have the same graph spectrum are called *cospectral* or *isospectral* graphs.
The four irregularity measures of interest in this study are presented next. The first one is based on the spectral radius of graph. If a graph G is regular, then it holds that the mean degree $\overline{d}(G)$ is equal to its spectral radius $\rho(G)$. Collatz and Sinogowitz [8] introduced the difference of these quantities as a measure of irregularity of G:

$$\mathrm{CS}(G) = \rho(G) - \overline{d}(G).$$

The first investigated irregularity measure that depends solely on the vertex degrees of a graph G is the *variance of the vertex degrees*, defined as

$$\mathrm{Var}(G) = \frac{1}{n} \sum_{i=1}^{n} d_G^2(v_i) - \frac{1}{n^2} \left(\sum_{i=1}^{n} d_G(v_i) \right)^2 .$$

Bell [6] has compared $\mathrm{CS}(G)$ and Var(G) and showed that they are not always compatible. Albertson [4] defines the *irregularity* of G as

$$\mathrm{irr}(G) = \sum_{uv \in E} \left| d_G(u) - d_G(v) \right| .$$

In [1] a new irregularity measure, related to the irregularity measure by Albertson was introduced. This measure also captures the irregularity only by the difference of vertex degrees. For a graph G, it is defined as

$$\mathrm{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} \left| d_G(u) - d_G(v) \right| .$$

Very recently, irr and irr_t were compared in [9].
These irregularity measures as well as other attempts to measure the irregularity of a graph were studied in several works [2,3,5,13–15]. It is interesting that the above four irregularity measures are not always compatible for some pairs of graphs. The main purpose of this paper is to determine classes of graphs for which two or more of the above mentioned irregularity measures are equal.
The rest of the paper is organized as follows: In Section 2 we investigate tridegreed graphs that have equal two or more of the above presented regularity measures. In Section 3 we consider the same problem but for graphs with arbitrary large degree sets. The smallest graphs with equal irregularity measures are investigated in Section 4. Final remarks and open problems are presented in Section 5.

2 Tridegreed graphs

Most of the results presented in this section are generalized in Section 3. However, due to the uniqueness of the related proofs and used constructions, we present the results of tridegreed graphs separately.

2.1 An infinite sequence of tridegreed graphs with same irr and irr_t indices

Proposition 1. *Let n be an arbitrary positive integer larger than 7. Then there exists a tridegreed graph with n vertices $J(n)$ for which* $\mathrm{irr}(J(n)) = \mathrm{irr}_t(J(n))$ *holds.*

Proof. The graph $J(n)$ can be constructed as $J(n) = C_{n-3} + P_3$, where C_{n-3} is a cycle on $n-3$ vertices and P_3 is a path on 3 vertices. It is easy to see that the graph obtained is tridegreed if n is larger than 7, and it contains one universal vertex, exactly. The vertex degree distribution of $J(n)$ is $n_5 = n-3$, $n_{n-2} = 2$ and $n_{n-1} = 1$. It can be shown that for $J(n)$ the equality $\mathrm{irr}(G) = \mathrm{irr}_t(G)$ holds. As an example graph $J(9)$ is depicted in Figure 1.
It is easy to show that for graph $J(9)$ the corresponding edge parameters are: $m_{5,5} = 6, m_{5,7} = 12, m_{5,8} = 6, m_{7,8} = 2$. Moreover, the equality $\mathrm{irr}(J(9)) = \mathrm{irr}_t(J(9)) = 44$ holds. □

2.2 Pairs of tridegreed graphs with same irr, irr_t and Var indices

Theorem 1. *Let G_a and G_b be connected edge-equivalent graphs. Then the equalities* $\mathrm{irr}(G_a) = \mathrm{irr}(G_b), \mathrm{irr}_t(G_a) = \mathrm{irr}_t(G_b)$ *and* $\mathrm{Var}(G_a) = \mathrm{Var}(G_b)$ *hold.*

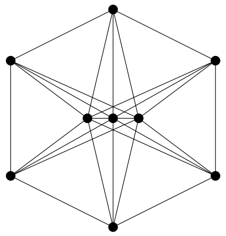


Fig. 1. The tridegreed graph $J(9)$

Proof. By definition, $\mathrm{irr}(G)$ depends solely on the edge parameters of G. Since graphs G_a and G_b have same edge parameters, it follows that $\mathrm{irr}(G_a) = \mathrm{irr}(G_b)$. From the definitions of $\mathrm{irr}_t(G)$ and $\mathrm{Var}(G)$ indices, we have

$$\mathrm{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d(u) - d(v)| = \sum_{r} \sum_{s<r} n_r n_s (r - s),$$

$$\mathrm{Var}(G) = \frac{1}{n} \sum_{u \in V(G)} d^2(u) - \left(\frac{2m}{n}\right)^2 = \frac{1}{n} \sum_{r} n_r \left(r - \frac{2m}{n}\right)^2.$$

So, $\mathrm{irr}_t(G)$ and $\mathrm{Var}(G)$ depend only on the vertex parameters of G. Since

$$r \cdot n_r(G) = \sum_{s \neq r} m_{r,s} + 2m_{r,r},$$

it follows that if graphs G_a and G_b are edge-equivalent, then G_a and G_b are necessarily vertex-equivalent as well, that is they have identical vertex-parameter set. Then, it also holds that $\mathrm{irr}_t(G_a) = \mathrm{irr}_t(G_b)$ and $\mathrm{Var}(G_a) = \mathrm{Var}(G_b)$. □

In Figure 2, two infinite sequences of pairs of tridegreed planar graphs that satisfied Theorem 1 are depicted. For a fixed integer $k \geq 1$, the graph $G_a(k)$ contains k hexagons, while the graph $G_b(k)$ contains k quadrangles. Graphs $G_a(k)$ and $G_b(k)$ have identical edge-parameters: $m_{3,1} = 4$, $m_{3,2} = 4k$, $m_{3,3} = k + 1$, and $n = 4k + 6$, $m = 5k + 5$. This implies that $G_a(k)$ and $G_b(k)$ have identical irregularity indices irr, irr_t and Var.
In what follows, we will verify that the converse of Theorem 1 is not true.

Proposition 2. *There exist tridegreed connected graphs with different edge-parameter distributions but identical* $\mathrm{irr}, \mathrm{irr}_t, \mathrm{Var}$ *and* CS *irregularity indices.*

Proof. An example is given in Figure 3. It is easy to see that polyhedral graphs (nanohedra graphs) depicted in Figure 3 are characterized by the following fundamental properties:

i) Polyhedral graphs G_c and G_d have $n = 8$ vertices and $m = 15$ edges.

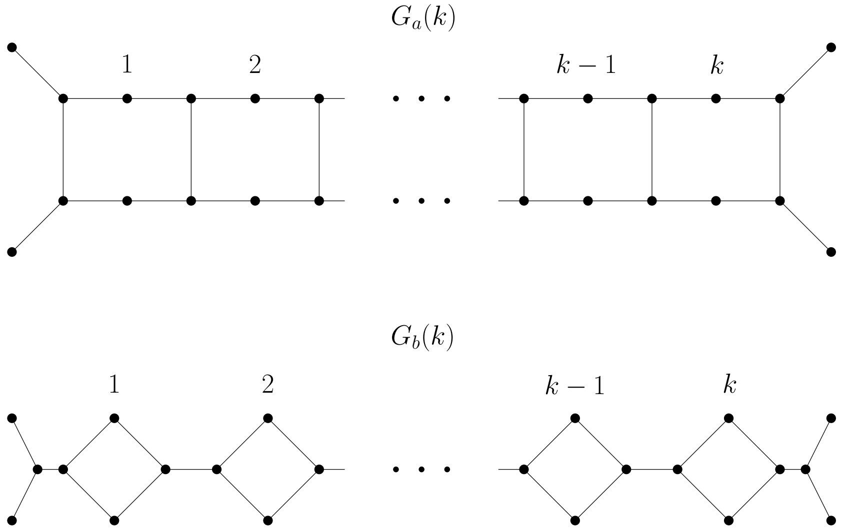


Fig. 2. Edge-equivalent graphs $G_a(k)$ and $G_a(k)$

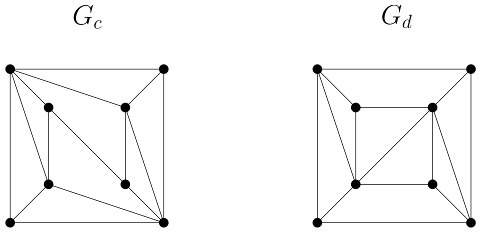


Fig. 3. Tridegreed polyhedral cospectral graphs [16] having identical degree sequence and different edge-parameter distribution, but identical $\mathrm{irr}, \mathrm{irr}_t$, Var and CS irregularity indices

ii) They have the same degree distribution: $n_3 = 4$, $n_4 = 2$ and $n_5 = 2$. This implies that $\mathrm{Var}(G_c) = \mathrm{Var}(G_d)$, and their total irregularity indices are equal, $\mathrm{irr}_t(G_c) = \mathrm{irr}_t(G_d)$.

iii) Their edge-parameter distributions are different, namely for graph $G_c(m_{33} = 1, m_{34} = 4, m_{35} = 6, m_{45} = 4)$ and for graph $G_d(m_{34} = 6, m_{35} = 6, m_{45} = 2, m_{55} = 1)$.

iv) Their Albertson indices are equal, $\mathrm{irr}(G_c) = \mathrm{irr}(G_d) = 20$. (This is an interesting fact, because the edge-parameter distributions of graphs G_c and G_d are different).

v) G_c and G_d are isospectral graphs (polyhedral twin graphs) [16] . This implies that their Collatz-Sinogowitz indices are equal, as well. □

2.3 Pairs of tridegreed graphs with same $\mathbf{irr}, \mathbf{irr}_t$, Var and CS indices

First, we state some necessary definitions and results needed for the derivation of the main results of this section. A bipartite graph G is *semiregular* if every edge of G joins a vertex of degree δ to a vertex of degree Δ. The 2*-degree* of a vertex u, denoted by $d_2(u)$ is the sum of degrees of the vertices adjacent to u [20]. The average-degree of u is $d_2(u)/d(u)$ and it is denoted by $p(u)$. A graph G is called *pseudo-regular* (or *harmonic*) if every vertex of G has equal average-degree. A bipartite graph is called *pseudo-semiregular* if each vertex in the same part of a bipartition has the same average-degree [20]. It follows that semiregular graphs form a subset of pseudo-semiregular graphs.

Theorem 2 ([20]). *Let G be a connected graph with degree sequence $(d_1, d_2, \ldots, d_n)$. Then*

$$\rho(G) \geq \sqrt{\frac{d_2(v_1)^2 + d_2(v_2)^2 + \cdots + d_2(v_n)^2}{d_1^2 + d_2^2 + \cdots + d_n^2}},$$

with equality if and only if G is a pseudo-regular graph or a pseudo-semiregular graph.

The following result is a consequence of Theorem 2.

Corollary 1 ([20]). *Let G be a pseudo-regular graph with $d_2(v) = p \cdot d(v)$ for each $v \in V(G)$, then $\rho(G) = p$.*

Theorem 3. *There are infinitely many pairs of tridegreed pseudo-regular graphs (G_1, G_2) for which $\mathrm{irr}(G_1) = \mathrm{irr}(G_2)$, $\mathrm{irr}_t(G_1) = \mathrm{irr}_t(G_2)$, $\mathrm{Var}(G_1) = \mathrm{Var}(G_2)$, and $\mathrm{CS}(G_1) = \mathrm{CS}(G_2)$.*

Proof. We prove the theorem by a construction. Let $G(2, x, y)$ be a graph with vertex set $V(G(2, x, y)) = U \cup W \cup \{z\}$ with connectivity determined as follows: vertex set $U = \{u_1, u_2, \ldots, u_x\}$ induces connected 2-regular subgraph (cycle $c_1 c_2 \cdot c_x$); W is comprised of $y \cdot x$ pendant vertices such that each vertex from U is adjacent to y vertices from W; and the 'central vertex' z is adjacent to each vertex from U. Two instances of such graphs, $G_1 = G(2, 7, 1)$ and $G_3 = G(2, 13, 2)$, are depicted in Figure 4. The parameter 2 in the graph's representation indicates that the vertex set U induces connected 2-regular subgraph. The average degree of a vertex from U is $(2(3+y)+x+y)/(3+y)$. The average degree of a vertex from W is $3+y$, which is also the average degree of z. Thus, $G(2, x, y)$ is pseudo-regular graph if $(2(3 + y) + x + y)/(3 + y) = (3 + y)$, or if

$$x = y^2 + 3y + 3. \tag{1}$$

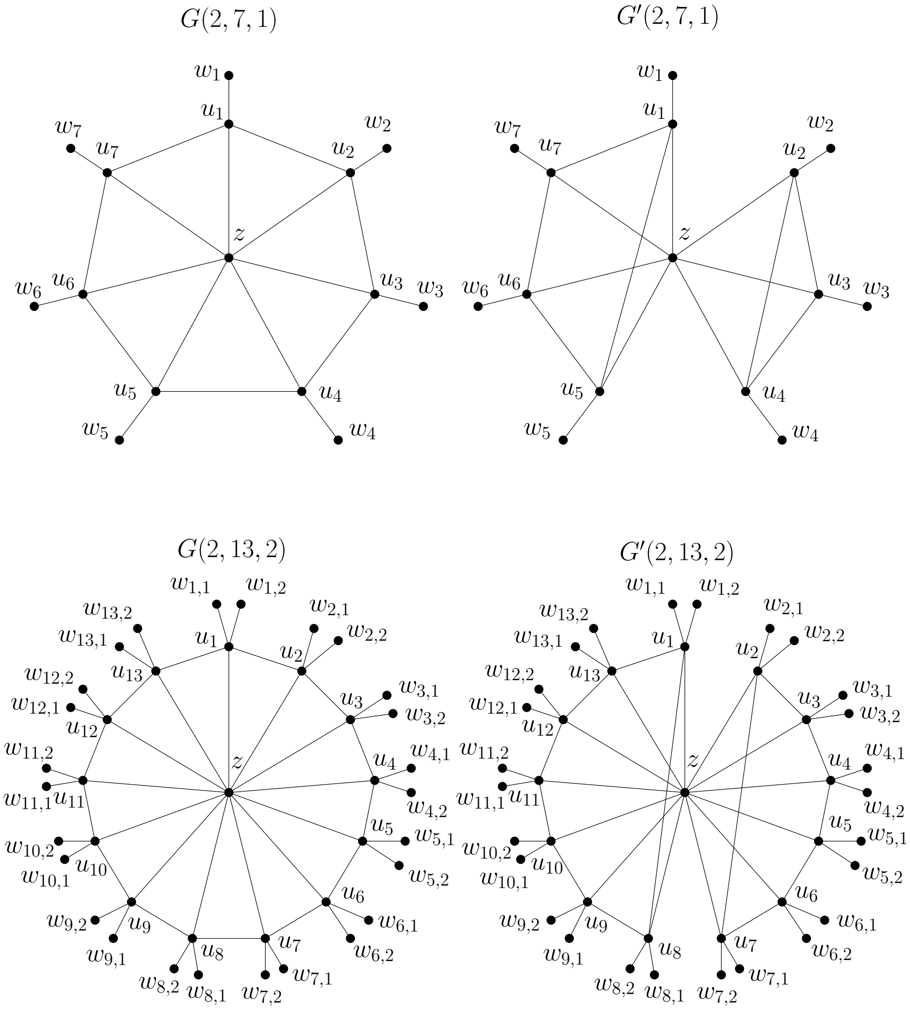


Fig. 4. Pseudo-regular graphs $G(2,7,1)$, $G'(2,7,1)$, $G(2,13,2)$, $G'(2,13,2)$, with $\rho(G(2,7,1)) = \rho(G'(2,7,1)) = 4$ and $\rho(G(2,13,2)) = \rho(G'(2,13,2)) = 5$

Next, consider a pair of edges (u_iu_{i+1}, u_ju_{j+1}) such that $(i \bmod x) + 2 < j$. We delete edges u_iu_{i+1} and u_ju_{j+1} and add edges u_iu_{j+1} and $u_{i+1}u_j$ to G_1, obtaining a graph $G'(2,x,y)$, which is edge equivalent (and therefore vertex equivalent) to $G(2,x,y)$. Also, the average degrees of the vertices of $G'(2,x,y)$ are equal to the average degrees of the vertices of $G(2,x,y)$. By Corollary 1, $\rho(G(2,x,y)) = \rho(G'(2,x,y)) = 3 + y$, for infinitely many integer solutions (x,y) of (1). This together with the fact that $G(2,x,y)$ and $G'(2,x,y)$ are edge equivalent, gives that $G(2,x,y)$ and $G'(2,x,y)$ have equal irr, irr_t, Var and CS indices. □

Two pairs of pseudo-regular graphs $G(2,7,1)$ and $G'(2,7,1)$, and pair $G(2,13,2)$ and $G'(2,13,2)$, for which Theorem 3 holds, are depicted in Figure 4. These graphs corresponds to the first two smallest pairs of integers that solve the equation (1).

Observe that the class of pair of graphs that satisfies Theorem 3 can be extended by considering graphs $G(k,x,y)$, $k \geq 2$. These graphs are generalization of $G(2,x,y)$ graphs, in such a way that the vertex set U induces a k-regular subgraph.

An alternative construction. Next, we will present a new construction, that asserts the claim of Theorem 3. This construction is based on so-called *Seidel switching* [19], which for a vertex v flips all the adjacency relationships with other vertices, i.e, all of the edges adjacent to v are removed and the edges that were not adjacent to v are added. In general, for a subset S of $V(G)$, the graph H is obtain from the graph G by *switching about* S if $V(H) = V(G)$ and $E(H) = \{uv \in E(G) | u,v \in S \text{ or } u,v \notin S\} \cup \{uv \notin E(G) | u \in S \text{ and } v \notin S\}$.

Construction by local switching.[[11]] Let G be a graph and let $\pi = (C_1, C_2, \ldots, C_k, D)$ be a partition of $V(G)$. Suppose that, whenever $1 \leq i,j \leq k$ and $v \in D$, we have

(a) any two vertices of C_i have same number of neighbors in C_j, and
(b) v has either 0, $n_i/2$ or n_i neighbors in C_i, where $n_i = |C_i|$.

The graph $G^{(\pi)}$ formed by *local switching in* G *with respect to* π is obtained from G as follows. For each $v \in D$ and $1 \leq i \leq k$ such that v has $n_i/2$ neighbors in C_i, delete these $n_i/2$ and join v instead to the other $n_i/2$ vertices in C_i.

The property of the above construction that will be used here is the following one.

Theorem 4 ([11]). *Let* G *be a graph and let* π *be a partition of* $V(G)$ *which satisfies properties* (a) *and* (b) *above. Then* $G^{(\pi)}$ *and* G *are cospectral, with cospectral compilements.*

The following construction is a special case of the construction by local switching, and will be used to construct infinite series of pairs of graph with the property stated in Theorem 3.

An example of the construction by local switching. A graph G is comprised of k-regular graph H on even number of vertices and one additional vertex v

adjacent to exactly half of the vertices of H. For $\pi(V(H), \{v\})$, we have that $G^{(\pi)}$ is obtained by joining v instead to the other vertices of H.

In the above example, as it was mentioned in [11], if H has $2m$ vertices and a trivial automorphism group, than all $\binom{2m}{m}$ possible realisations of H are non-isomorphic. By Theorem 4 the graphs G and $G^{(\pi)}$ are cospectral. G and $G^{(\pi)}$ have also same degree set $\mathcal{D}(G) = \mathcal{D}(G^{(\pi)}) = \{k, k+1, m\}$. The number of edges with endvertices with degrees m and k in G is the same as in $G^{(\pi)}$. The same holds for edges with endvertices with degrees m and $k+1$, and m and $m+1$. Thus, G and $G^{(\pi)}$ are edge equivalent, and $\mathrm{irr}(G) = \mathrm{irr}(G^{(\pi)})$, $\mathrm{irr}_t(G) = \mathrm{irr}_t(G^{(\pi)})$,$\mathrm{Var}(G) = \mathrm{Var}(G^{(\pi)})$ and $\mathrm{CS}(G) = \mathrm{CS}(G^{(\pi)})$. Note that if H has less than 8 vertices, then G and $G^{(\pi)}$ are isomorphic. In Figure 5 an example of Seidel switching for $H = C_8$ (cycle with 8 vertices) is depicted.

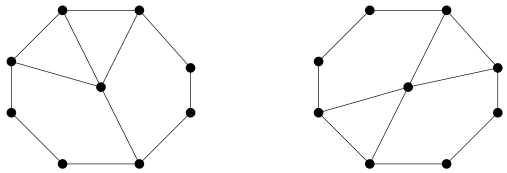


Fig. 5. Seidel switching when H is a cycle with 8 vertices

3 Graphs with arbitrary large degree set and same irregularity indices

3.1 An infinite sequence of graphs with same irr and irr_t indices

A graph G is a *complete k-partite* graph if there is a partiton $V_1 \cup \cdots \cup V_k = V(G)$ of the vertex set, such that $uv \in E(G)$ if and only if u and v are in different parts of the partition.

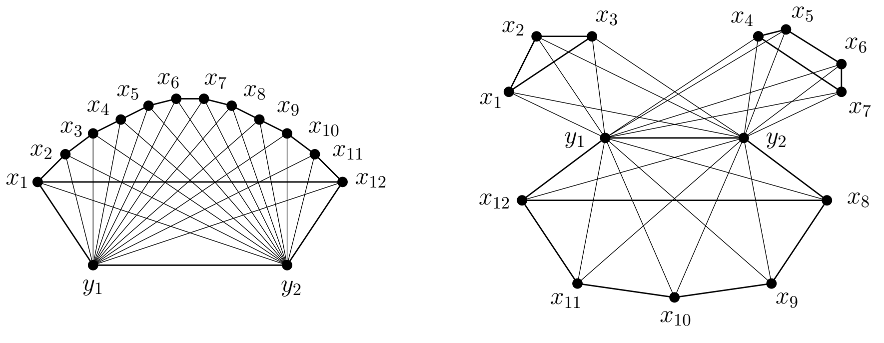
Proposition 3. *There is an infinite sequences of graphs $\mathcal{G}$, such that for a graph $G \in \mathcal{G}$ $\mathrm{irr}(G) = \mathrm{irr}_t(G)$ holds.*

Proof. If every two vertices of G with different degrees are adjacent, then $\mathrm{irr}(G) = \mathrm{irr}_t(G)$. Graphs that satisfy this condition are the complete k-partite graphs. □

3.2 Pairs of graphs with arbitrary large degree set and same irr, irr_t, and Var indices

Proposition 4. *There are infinitely many graphs G_1 and G_2 with same arbitrary cardinality of their degree sets satisfying* $\mathrm{irr}(G_1) = \mathrm{irr}(G_2)$, $\mathrm{irr}_t(G_1) = \mathrm{irr}_t(G_2)$, *and* $\mathrm{Var}(G_1) = \mathrm{Var}(G_2)$.

Proof. Consider the graphs $G^1_{csl}(14,2,4)$ and $G^2_{csl}(14,2,4)$ depicted in Figure 6. The graphs are bidegreed edge-equivalent, belong to the so-called *complete split-like* graphs, and were introduced and studied in [17]. Choose vertices $u \in V(G^1_{csl}(14,2,4))$ and $u \in V(G^2_{csl}(14,2,4))$ such that $d(u) = d(v)$. Attach to u an arbitrary graph H obtaining a graph G_1. Attach to v a copy of H obtaining a graph G_2. The graphs G_1 and G_2 are also edge-equivalent and therefore, $\mathrm{irr}(G_1) = \mathrm{irr}(G_2)$, $\mathrm{irr}_t(G_1) = \mathrm{irr}_t(G_2)$, $\mathrm{Var}(G_1) = \mathrm{Var}(G_2)$. □



(a) (b)

Fig. 6. Bidegreed edge-equivalent complete split-like graphs, (a) $G^1_{csl}(14,2,4)$ and (b) $G^2_{csl}(14,2,4)$

Observe that in the construction, presented in the above proof, one instead of $G^1_{csl}(14,2,4)$ and $G^2_{csl}(14,2,4)$ can use any edge-equivalent graphs, for example graphs $G_a(k)$ and $G_b(k)$ in Figure 2.

3.3 Pairs of graphs with arbitrary large degree set and same $\mathbf{irr}, \mathbf{irr}_t$, Var and CS indices

The 0-*sum* of two graphs G and H is got by identifying a vertex in G with a vertex in H. To obtain the result of this section, we will use the following theorem and a corollary of it.

Theorem 5 ([10]). *Let F be a* 0*-sum obtained by merging v in G with v in H, then the characteristic polynomial of F is*

$$\phi(F,\lambda) = \phi(G,\lambda)\phi(H \setminus v,\lambda) + \phi(G \setminus v,\lambda)\phi(H,\lambda) - \lambda\phi(G \setminus v,\lambda)\phi(H \setminus v,\lambda).$$

Corollary 2 ([10]). *If we hold G and its vertex v fixed, then the characteristic polynomial of the* 0*-sum of G and H is determined by the characteristic polynomials of H and $H \setminus v$.*

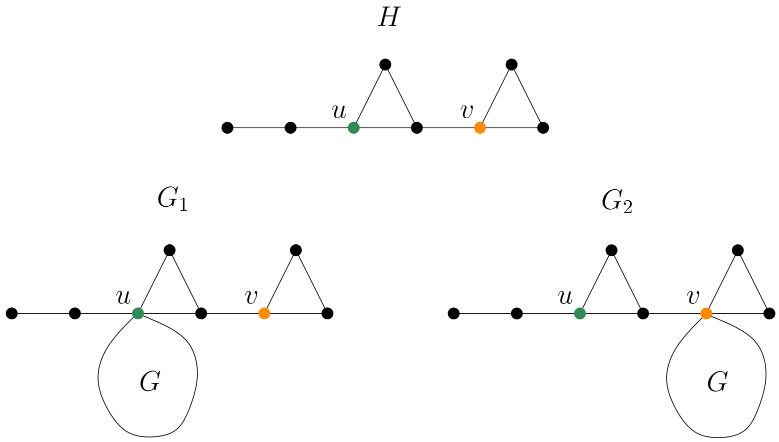


Fig. 7. Two cospectral and edge-equvivalent graphs G_1 and G_2 obtained as 0-sums of H and arbitrary graph G

Theorem 6. *There are infinitely many graphs G_1 and G_2 with same arbitrary cardinality of their degree sets satisfying* $\mathrm{irr}(G_1) = \mathrm{irr}(G_2)$, $\mathrm{irr}_t(G_1) = \mathrm{irr}_t(G_2)$, $\mathrm{Var}(G_1) = \mathrm{Var}(G_2)$, *and* $\mathrm{CS}(G_1) = \mathrm{CS}(G_2)$.

Proof. Let G be an arbitrary graph. Consider the graph H in Figure 7. Let G_1 be a 0-sum of H and G, obtained by merging v in G with v in H, and G_2 be a 0-sum obtained by merging u in G with u in H. Note that $H \setminus v$ and $H \setminus u$ are isomorphic, so $\phi(H \setminus v, \lambda) = \phi(H \setminus u, \lambda)$. Together with Corollary 2, we have that $\phi(G_1, \lambda) = \phi(G_1, \lambda)$, or that G_1 and G_2 are cospectral. Also, it is easy to see that G_1 and G_2 are edge-equivalent. Thus, G_1 and G_2 have same irr, irr_t, Var and CS indices. ☐

A generalization of the example from Figure 7 is given in Figure 8. The graph H is comprised of three isomorphic subgraphs Q_l, Q_m, Q_r, each of order at least 3, and two vertices u and v. Between the vertex v and the subgraph Q_r, there are same number of edges as between the vertex u and the subgraph Q_m. Also, between the vertex u and the subgraph Q_l, there are same number of edges as between the vertex v and the subgraph Q_m. The number of the edges between v and subgraph Q_m differs than the number of the edges between v and subgraph Q_r. We require these conditions to avoid an isomorphism of graphs G_1 and G_2, obtained as 0-sums of H and arbitrary graph G. The graphs G_1 and G_2 are constructed in the same manner as above: G_1 is a 0-sum obtained by merging v in G with v in H, and G_2 be a 0-sum obtained by merging u in G with u in H. From the construction it follows that G_1 and G_2 are edge-equivalent. In this case also $H \setminus v$ and $H \setminus u$ are isomorphic, so $\phi(H \setminus v, \lambda) = \phi(H \setminus u, \lambda)$. Together with Corollary 2, we have that $\phi(G_1, \lambda) = \phi(G_1, \lambda)$, or that G_1 and G_2 are cospectral. Thus, it

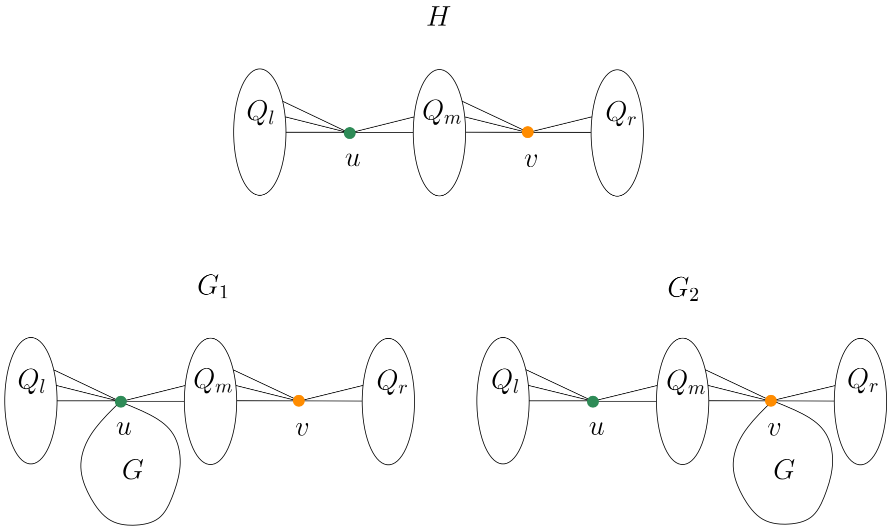


Fig. 8. A generalization of the example from Figure 7

holds that $\mathrm{irr}(G_1) = \mathrm{irr}(G_2)$, $\mathrm{irr}_t(G_1) = \mathrm{irr}_t(G_2)$, $\mathrm{Var}(G_1) = \mathrm{Var}(G_2)$, and $\mathrm{CS}(G_1) = \mathrm{CS}(G_2)$.

4 Small graphs with identical irregularities

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We said that G_1 *is smaller than* G_2 if and only if $|V_1| + |E_1| < |V_2| + |E_2|$. Consequently, for two pairs of graphs $P_1 = (G_1, G_2)$ and $P_2 = (G_3, G_4)$, we said that P_1 *is smaller than* P_2 if and only if $|V_1| + |E_1| + |V_2| + |E_2| < |V_3| + |E_3| + |V_4| + |E_4|$. The results in this section are obtained by computer search using mathematical software package Sage [18].

Proposition 5. *There are no two graphs, both of same order* $n \leq 5$*, that have identical irregularity indices* $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ *and* irr_t*.*

Next the smallest example of pair of graphs will be given with equal $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t indices.

4.1 Graphs of order 6

The smallest pair of graphs with identical irregularity indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t is depicted in Figure 9. Both graphs are of order 6, but one is of size 6 and the other of size 9. Their $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t indices are 0.236068, 0.800000, 8, and 16, respectively. They have different spectral radii, namely

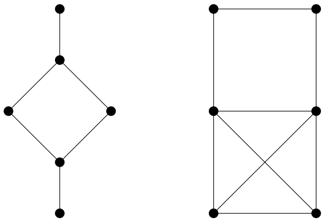


Fig. 9. The smallest pair of (tridegreed) connected graphs with identical irregularity indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t

the smaller one has spectral radius 2.236068 and bigger one 3.236068. The rest of the graphs of order 6, with identical irregularity indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t are given in Figure 10. The parameters of the graphs of order 6 with identical irregularity indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t are summarized in Table 1. The graphs are enumerated with respect to their sizes, a smaller graph has smaller associated number (G_no). For a given graph, beside the values of the

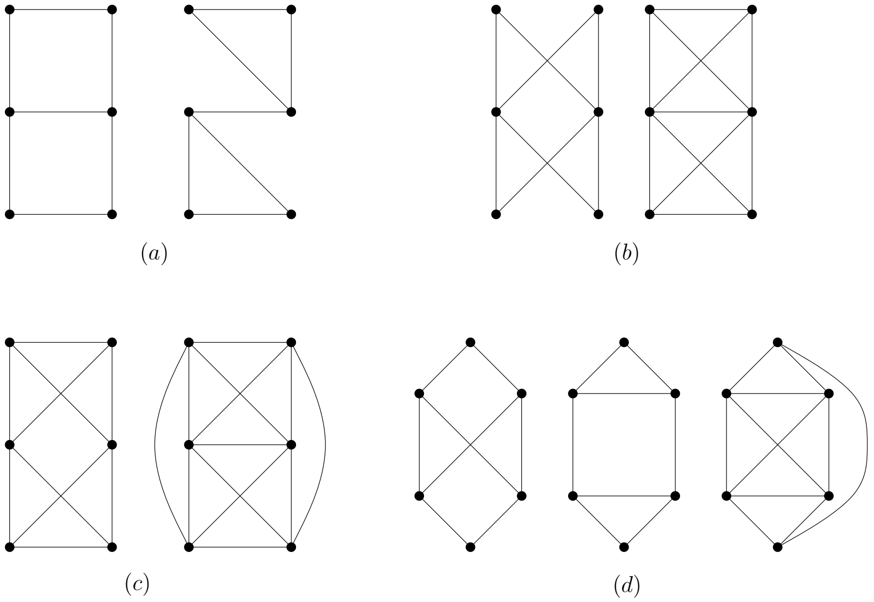


Fig. 10. Besides the pair in Figure 9, there are three other pairs of connected graphs of order 6 (a), (b), (c), and only one triple of graphs of order 6 (d) with identical irregularity indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t

indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t, its spectral radius ρ, degree sequence and graph6 code are given. There are 112 non-isomorphic connected graphs of order 6.

Table 1: All four pairs and the only triple of graphs of order 6 with identical irregularity indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t

tuple_no	G_no	graph6	degree sequence	irr	irr_t	CS	Var	ρ
1	10	E?]o	[3, 3, 2, 2, 1, 1]	8	16	0.236068	0.800000	2.236068
	77	E‘]w	[4, 4, 3, 3, 2, 2]	8	16	0.236068	0.800000	3.236068
2	36	EKNG	[3, 3, 2, 2, 2, 2]	4	8	0.080880	0.266667	2.414214
	37	E‘NG	[3, 3, 2, 2, 2, 2]	4	8	0.080880	0.266667	2.414214
3	40	E?∼o	[4, 4, 2, 2, 2, 2]	16	16	0.161760	1.066667	2.828427
	100	EK∼w	[5, 5, 3, 3, 3, 3]	16	16	0.161760	1.066667	3.828427
4	90	EK∼o	[4, 4, 3, 3, 3, 3]	8	8	0.038948	0.266667	3.372281
	110	E]∼w	[5, 5, 4, 4, 4, 4]	8	8	0.038948	0.266667	4.372281
5	54	EImo	[3, 3, 3, 3, 2, 2]	4	8	0.065384	0.266667	2.732051
	55	EJeg	[3, 3, 3, 3, 2, 2]	4	8	0.065384	0.266667	2.732051
	103	Ejmw	[4, 4, 4, 4, 3, 3]	4	8	0.065384	0.266667	3.732051

4.2 Graphs of order 7

There are 853 non-isomorphic connected graphs of order 7. The pairs of graphs of order 7 with identical irregularity indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t are given in Table 2. The smallest pair of connected graphs of order 7 with identical irregularity indices is depicted in Figure 11.

Table 2: All pairs of graphs of order 7 with identical irregularity indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t.

pair_no	G_no	graph6	degree sequence	irr	irr_t	CS	Var	ρ
1	104	FK?}O	[3, 3, 2, 2, 2, 2, 2]	6	10	0.057209	0.238095	2.342923
	105	F‘?}O	[3, 3, 2, 2, 2, 2, 2]	6	10	0.057209	0.238095	2.342923
2	177	FAerO	[3, 3, 3, 3, 2, 2, 2]	6	12	0.069758	0.285714	2.641186
	178	FAdtO	[3, 3, 3, 3, 2, 2, 2]	6	12	0.069758	0.285714	2.641186
3	213	FK?}W	[4, 4, 2, 2, 2, 2, 2]	12	20	0.242178	0.952381	2.813607
	214	F‘?}W	[4, 4, 2, 2, 2, 2, 2]	12	20	0.242178	0.952381	2.813607
4	244	F?v‘w	[4, 4, 4, 2, 2, 2, 2]	16	24	0.217713	1.14285	3.074856
	269	FA_∼o	[5, 3, 3, 3, 2, 2, 2]	16	24	0.217713	1.142857	3.074856
5	274	F@VTW	[4, 4, 3, 3, 2, 2, 2]	12	22	0.173179	0.809524	3.030322
	275	F@UuW	[4, 4, 3, 3, 2, 2, 2]	12	22	0.173179	0.809524	3.030322

6	321	FIe`w	[4, 3, 3, 3, 3, 2, 2]	8	16	0.142857	0.476190	3.000000
	322	FJaHw	[4, 3, 3, 3, 3, 2, 2]	8	16	0.142857	0.476190	3.000000
7	348	FBY^?	[3, 3, 3, 3, 3, 3, 2]	2	6	0.046069	0.142857	2.903212
	349	FHU^?	[3, 3, 3, 3, 3, 3, 2]	2	6	0.046069	0.142857	2.903212
8	471	Fie`w	[4, 3, 3, 3, 3, 3, 3]	4	6	0.034553	0.142857	3.177410
	472	FjaHw	[4, 3, 3, 3, 3, 3, 3]	4	6	0.034553	0.142857	3.177410
9	438	FBY^G	[4, 4, 3, 3, 3, 3, 2]	8	16	0.093211	0.476190	3.236068
	439	FHU^G	[4, 4, 3, 3, 3, 3, 2]	8	16	0.093211	0.476190	3.236068
10	450	FKMiw	[4, 4, 4, 3, 3, 2, 2]	10	22	0.210999	0.809524	3.353856
	451	FKLkw	[4, 4, 4, 3, 3, 2, 2]	10	22	0.210999	0.809524	3.353856
11	444	FHU[w	[4, 4, 4, 4, 2, 2, 2]	8	24	0.346431	1.142857	3.489289
	469	FKWyw	[4, 4, 4, 3, 3, 3, 1]	8	24	0.346431	1.142857	3.489289
12	500	F?]}w	[5, 5, 5, 3, 2, 2, 2]	24	36	0.399856	2.285714	3.828427
	536	F@U^w	[6, 4, 4, 4, 2, 2, 2]	24	36	0.399856	2.285714	3.828427
13	543	FBY|o	[4, 4, 4, 4, 4, 2, 2]	8	20	0.217180	0.952381	3.645751
	544	FB]lg	[4, 4, 4, 4, 4, 2, 2]	8	20	0.217180	0.952381	3.645751
14	604	FIefw	[6, 3, 3, 3, 3, 3, 3]	18	18	0.217180	1.285714	3.645751
	613	FJaNw	[6, 3, 3, 3, 3, 3, 3]	18	18	0.217180	1.285714	3.645751
15	609	FkUhw	[4, 4, 4, 3, 3, 3, 3]	6	12	0.074653	0.285714	3.503224
	610	FkYXw	[4, 4, 4, 3, 3, 3, 3]	6	12	0.074653	0.285714	3.503224
16	656	FG]}w	[5, 5, 5, 3, 3, 3, 2]	20	30	0.268873	1.571429	3.983159
	665	FHU^w	[6, 4, 4, 4, 3, 3, 2]	20	30	0.268873	1.571429	3.983159
17	695	FKNNw	[6, 4, 4, 3, 3, 3, 3]	20	24	0.203000	1.238095	3.917286
	697	F`NNw	[6, 4, 4, 3, 3, 3, 3]	20	24	0.203000	1.238095	3.917286
18	701	FbY|o	[4, 4, 4, 4, 4, 3, 3]	4	10	0.064171	0.238095	3.778457
	702	Fb]lg	[4, 4, 4, 4, 4, 3, 3]	4	10	0.064171	0.238095	3.778457
19	748	FImvw	[6, 4, 4, 4, 4, 3, 3]	18	22	0.156325	1.000000	4.156325
	750	FJenw	[6, 4, 4, 4, 4, 3, 3]	18	22	0.156325	1.000000	4.156325
20	501	F?∼v_	[4, 4, 4, 3, 3, 3, 3]	12	12	0.035530	0.285714	3.464102
	851	F]∼∼w	[6, 6, 6, 5, 5, 5, 5]	12	12	0.035530	0.285714	5.464102
21	810	FFzfw	[6, 4, 4, 4, 4, 4, 4]	12	12	0.086567	0.571429	4.372281
	812	FLvfw	[6, 4, 4, 4, 4, 4, 4]	12	12	0.086567	0.571429	4.372281

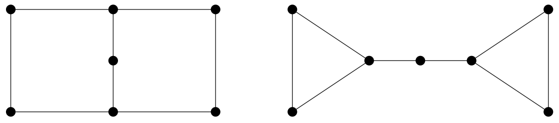


Fig. 11. The smallest pair of connected graphs of order 7 with identical irregularity indices CS, Var, irr and irr_t (the pair 1 in Table 2)

We would like to note that there is no triple of graphs of order 7 with identical irregularity indices $\mathrm{CS}, \mathrm{Var}, \mathrm{irr}$ and irr_t.

5 Conclusion and open problems

We have studied four established measures of irregularity of a graph. In particular, we have considered the problem of determining pairs or classes of graphs for which two or more of the purposed measures are equal. Some related results in the case of bidegreed graphs were presented in [17]. Here we have extended that work for tridegreed graphs and graphs with arbitrarily large degree set.

In the investigations here, it was assumed that considered graphs are of the same order, or they even have same degree sets. With respect to that, there are several interesting extension of the work done here.

It would be of interest to determine graphs of same order which have different degree sets, but their corresponding irr_t and irr indices are identical. A graph pair of such type with 5 vertices is illustrated in Figure 12. Also, it would be of interested to find classes of graphs of different order with equal irregularity measures. Most of the result presented have involved only pairs of graphs. Extending those results to larger classes of graphs seems to be demanding but interesting problem. Finally, considering other irregularity measures could offer new insights in the topic.

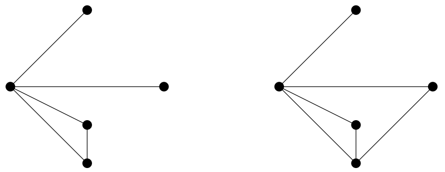


Fig. 12. A tridegreed and a four degreed planar graphs with identical $\mathrm{irr}_t = 14$ and $\mathrm{irr} = 10$ irregularity indices

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