Level Crossing Probabilities of the Ornstein – Uhlenbeck Process

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Abstract: The Ornstein Uhlenbeck process is a Gaussian process X_t *with independent increments and autocorrelation* $E(X_t X_{t+s}) = \frac{e^{-|s|}}{2}$. *First the Laplace transform of the probability density* $P(X_t = x | X_0 = p)$ *is computed. Using this, the Laplace transform of* X_t *first time reaching a given value* x *is derived. It is proved that these results agree with the special case derived earlier by Bellman and Harris (Pacific J. Math. 1, 1951).*

1 Definitons

The Ornstein Uhlenbeck process is a stationary Gaussian-Markov process X_t such that the joint distribution of $X_{t1,} X_{t2} \dots X_{tm}$ is a gaussian and is dependent only on the differences $t_j - t_i$ where $i < j$ and the autocorelation function is given by

$$E(X_s \cdot X_{s+t}) = \frac{1}{2} e^{-|t|} \tag{1.1}$$

$$EX_t = 0 \text{ and } EX_t^2 = \frac{1}{2}. \tag{1.2}$$

Let X be a random vector with normal distribution, then the density of its probability distribution is:

$$\frac{1}{2\pi|\Sigma|} e^{-\frac{1}{2} X^T \Sigma^{-1} X}$$

where $X = \begin{pmatrix} X \\ Y \end{pmatrix}$ and Σ is the correlation matrix:

$$\begin{pmatrix} \rho_1 & \sigma \\ \sigma & \rho_2 \end{pmatrix}$$

with $\rho_1 = EX^2, \rho_2 = EY^2, \sigma_1 = EXY$ and $|\Sigma| = \rho_1\rho_2 - \sigma^2$. Clearly

$$\Sigma^{-1} = \frac{\begin{pmatrix} \rho_2 & -\sigma \\ -\sigma & \rho_1 \end{pmatrix}}{\rho_1\rho_2 - \sigma^2}.$$

Hence the joint probability density

$$P(X = x, Y = y) = \frac{1}{2\pi\sqrt{\rho_1\rho_2 - \sigma^2}} \exp\left(-\frac{\rho_2 x^2 - 2\sigma xy + \rho_1 y^2}{2(\rho_1\rho_2 - \sigma^2)} \right).$$

It follows from here that

$$P(Y = y | X = x) = \frac{\frac{1}{2\pi\sqrt{\rho_1\rho_2 - \sigma^2}} \exp\left(-\frac{\rho_2 x^2 - 2\sigma xy + \rho_1 y^2}{2(\rho_1\rho_2 - \sigma^2)} \right)}{\frac{e^{-\frac{x^2}{2\rho_1}}}{\sqrt{2\pi\rho_1}}}$$

$$= \frac{1}{\sqrt{2\pi\frac{\rho_1\rho_2 - \sigma^2}{\rho_1}}} \exp\left(-\frac{\left(y - \frac{\sigma^2}{\rho_1} x \right)^2}{2\frac{\rho_1\rho_2 - \sigma^2}{\rho_1}} \right).$$

Applying this to what concerns us, the Ornstein-Uhlembeck process, we can determine the probability density $P(X_t = x \mid X_0 = p)$.

Clearly

$$\rho_1 = \rho_2 = \frac{1}{2}, \sigma = \frac{e^{-t}}{2} \text{ so } \frac{2(\rho_1\rho_2 - \sigma^2)}{\rho_1} = \frac{2\left(\frac{1}{2}\frac{1}{2} - \frac{e^{-2t}}{4}\right)}{\frac{1}{2}} = 1 - e^{-2t}. \ \frac{\sigma}{\rho_1} = e^{-t}$$

Hence:

$$P(X_t = x \mid X_0 = p) = \frac{e^{-\frac{\left(x - pe^{-t}\right)^2}{\left(1 - e^{-2t}\right)}}}{\sqrt{\pi\left(1 - e^{-2t}\right)}}. \tag{1.3}$$

We shall denote this with $P(t,p,x)$ or $P(p,x)$ and call it the fundamental function. The special cases $p = 0$ and $x = 0$ are important also:

$$P(X_t = x \mid X_0 = 0) = \frac{e^{-\frac{x^2}{\left(1 - e^{-2t}\right)}}}{\sqrt{\pi\left(1 - e^{-2t}\right)}}. \tag{1.4}$$

$$P(X_t = 0 \mid X_0 = p) = \frac{e^{-\frac{p^2 e^{-2t}}{\left(1 - e^{-2t}\right)}}}{\sqrt{\pi\left(1 - e^{-2t}\right)}}. \tag{1.5}$$

By simple substitution it is easy to prove that (1.3) satisfies the forward equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial t} + u$$

and the backward equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial p^2} - p\frac{\partial u}{\partial p}.$$

This also implies that (1.4) satisfies the forward equation and (1.5) satisfies the backward equation.

2 The Laplace Transforms of the Fundamental Functions

Since both $\frac{e^{-\frac{\left(x-pe^{-t}\right)^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi\left(1-e^{-2t}\right)}}$ and $\frac{e^{-\frac{x^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi\left(1-e^{-2t}\right)}}$ satisfy the $u_t = \frac{1}{2}u_{xx} + u + xu_x$ forward equation their Laplace transforms must satisfy the $sU = \frac{U_{xx}}{2} + U + xU_x$ second order ordinary differential equation, that is the equation

$$U'' + 2xU' + 2(1-s)U = 0 \tag{2.1}$$

To find the solutions of (2.1) let us consider the confluent hypergeometric equation

$$xy'' = +(c-x)y'0 - ay = 0 \tag{2.2}$$

The two solutions of this are the:

$$ {}_1F_1\left(a,c;x\right) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a\left(a+1\right)x^2}{c(c+1)2!}\dots $$

and $x^{1-c}\,{}_1F_1\left(a+1-c,2-c;x\right)$ Kummer functions. Let us consider the following transformation of (2.2) $u = y(kx^2)$ where k is an arbitrary nonzero constant.

Clearly:

$$
\begin{aligned}
&u = y(kx^2) \\
&u' = 2kxy' \\
&u'' = 2ky' + 4k^2x^2y''.
\end{aligned}
$$

Hence:

$$y = u$$
$$y' = \frac{u'}{2kx}$$
$$y'' = \frac{u'' - \frac{u'}{x}}{4k^2x^2}$$

Substituting these into (2.2) gives:

$$\frac{kx^2\left(u'' - \frac{u'}{x}\right)}{4k^2x^2} + (c - kx^2)\frac{u'}{2kx} - au = 0$$

which in turn, after some simplification becomes:

$$u'' + \left(\frac{2c-1}{x} - 2kx\right)u' - 4kau = 0.$$

Putting $c\frac{1}{2}$ gives: $u'' - 2kxu' - 4kau = 0.$

Let us compare this with (2.1)

$$U'' + 2xU' + 2(1-s)U = 0$$
$$-2k = 2$$
$$-4ka = 2(1-s).$$

Hence we get for k and for $ak = -1$ and $a = \frac{1-s}{2}$. Therefore the solutions of (2.1) are $F_1 = F\left(\frac{1-s}{2}, \frac{1}{2}; -x^2\right)$ and $F_2 = xF\left(1 - \frac{s}{2}, \frac{3}{2}; - - x^2\right)$.

Now we are in the position to determine the Laplace transform of $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$.

Clearly it must be of the form $AF_1 + xBF_2$ where A and B some constans. To this end Laplace transform will be evaluated for some special cases. The Laplace transform of $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ is clearly:

$$\int_0^\infty \frac{e^{-\frac{x^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)} e^{-st} dt.$$

Writing t instead of e^{-t} transforms it into a Mellin type integral:

$$\int_0^1 \frac{e^{-\frac{x^2}{1-t^2}}}{\sqrt{\pi}\left(1-t^2\right)} t^{s-1} dt.$$

Substituing $\sqrt{t}$ instead of t yields

$$\frac{1}{2\sqrt{\pi}} \int_0^1 \frac{e^{-\frac{x^2}{1-t}}}{\sqrt{1-t}} t^{\frac{s}{2}-1} dt.$$

For $x = 0$ this becomes the beta funciton type integral:

$$\frac{1}{2\sqrt{\pi}} \int_0^1 \frac{t^{\frac{s}{2}-1}}{\sqrt{1-t}} dt = \frac{1}{2\sqrt{\pi}} B\left(\frac{1}{2}, \frac{s}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{1+s}{2}\right)}.$$

Hence

$$A = \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)}.$$

Clearly A is the Laplace transform of $\frac{e^{-\frac{x^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$. To determine the value of B let us consider the x derivative of the Laplace transform, which is:

$$-\int_0^1 \frac{x e^{-\frac{x^2}{1-t}}}{\sqrt{\pi}(1-t)^{\frac{3}{2}}} t^{\frac{s}{2}-1} dt.$$

In the present case we cannot take the $x \to 0$ limit by simply substituing $x \to 0$ for x because $\frac{xe^{-\frac{x^2}{t}}}{\sqrt{\pi} - t^{\frac{3}{2}}}$ does not converge uniformly to 0 as $x \to 0$. In fact it is a "delta function type function", its integral being

$$\int_0^1 \frac{xe^{-\frac{x^2}{t}}}{\sqrt{\pi}t^{\frac{3}{2}}} dt = 1.$$

For it is know from theory of the heat equation that, for an arbitrary continous function $f(t)$

$$\lim_{x\to 0} \int_0^t \frac{x}{\sqrt{\pi}} \frac{e^{-\frac{x^2}{t-r}}}{(t-r)^{\frac{3}{2}}} f(r)dr = \lim_{x\to 0} \int_0^t \frac{x}{\sqrt{\pi}} \frac{e^{-\frac{x^2}{r}}}{r^{\frac{3}{2}}} f(t-r)dr = f(t).$$

Hence in the present case:

$$-\lim_{x\to 0} \int_0^1 \frac{xe^{-\frac{x^2}{1-t}}}{\sqrt{\pi}(1-t)^{\frac{3}{2}}} t^{\frac{s}{2}-1} dt = t^{\frac{s}{2}-1}\Big|_{t=1} = -1,$$

thus $B = -1$. Therefore the The Laplace transform of $\frac{e^{-\frac{x^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ is

$$AF_1 - xF_2 = \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)} F\left(\frac{1-s}{2}, \frac{1}{2}; -x^2\right) - xF\left(1-\frac{s}{2}, \frac{3}{2}; -x^2\right).$$

Now we compute the The Laplace transform of $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$. It has been shown that it statisfies the backward equation $u_t = -pu_p + \frac{u_{pp}}{2}$. Therefore its Laplace transform is the solution of the second order linear differential equation $sU = -pU_p + \frac{U_{pp}}{2}$ that is of the equation

$$U'' + 2pU' + 2sU = 0$$

Now the solution of $u'' - 2kxu' - 4kau = 0$ are $F\left(a, \frac{1}{2}; kx^2\right)$ and $xF\left(a + \frac{1}{2}, \frac{3}{2}; kx^2\right)$.

Comparing the two equations we get for k

$$2k = 2$$
$$4ka = 2s$$

that is $k = 1$ and $a = \frac{s}{2}$. Thus the Laplace transform must be the linear combination of $G_1 = F\left(\frac{s}{2}, \frac{1}{2}; p^2\right)$ and $pG_2 = pF\left(\frac{1+s}{2}, \frac{3}{2}; p^2\right)$. To find the conficciens of G_1 and pG_2 let us inspect the Laplace transform itself. $\int_0^\infty \frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}\left(1-e^{-2t}\right)} e^{-st} dt.$

Writing t instead of e^{-t} it transforms again into the Mellin type integral:

$$\int_0^1 \frac{e^{-\frac{p^2t^2}{(1-t^2)}}}{\sqrt{\pi}\left(1-t^2\right)} t^{s-1} dt.$$

Substituing $\sqrt{t}$ instead of t yields

$$\frac{1}{2\sqrt{\pi}}\int_0^1 \frac{e^{-\frac{p^2t^2}{1-t}}}{\sqrt{1-t}}\, t^{\frac{s}{2}-1}\,dt.$$

Again putting $p = 0$ this becomes:

$$\frac{1}{2\sqrt{\pi}}\int_0^1 \frac{t^{\frac{s}{2}}}{\sqrt{1-t}}\,dt = A.$$

The coefficient of pG_2 can be evaluated the same way as was done for $\frac{e^{-\frac{x^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ and it is found to be again –1. Thus the Laplace transform of $\frac{e^{-\frac{x^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ is

$$AG_1 - pG_2 = AF\left(\frac{s}{2},\frac{1}{2};p^2\right) - pF\left(1-\frac{1+s}{2},\frac{3}{2};p^2\right).$$

The above result can be arrived at directly from the Laplace transform of $\frac{e^{-\frac{x^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$.

To this end let us inspect

$$\int_0^\infty \frac{e^{-\frac{p^2e^{-t2}}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}\, e^{-st}\,dt$$

using

$$\frac{p^2e^{-2t}}{1-e^{-2t}} = \frac{p^2}{1-e^{-2t}} - p^2.$$

This becomes $e^{p^2}\int_0^\infty \frac{e^{-\frac{p^2}{(1-e^{-2t})}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}e^{-st}dt$ and the integra here is of the same form as of the Laplace transform of $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ except we have p instead of x.

Therfore the Laplace transform of $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ is

$$e^{p^2}\left(\frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)}F\left(\frac{1-s}{2},\frac{1}{2};-p^2\right)-pF\left(1-\frac{s}{2},\frac{3}{2};-p^2\right)\right)$$

Applying Kummer's formula $F(a,c;x)=e^xF(c-a,c;x)$ we get for the Laplace transform of $\frac{e^{-\frac{p^2e^{-2t}}{(1-e^{-2t})}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$

$$\frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)}F\left(\frac{s}{2},\frac{1}{2};p^2\right)-pF\left(1-\frac{1+s}{2},\frac{3}{2};p^2\right).$$

3 The Laplace Transforms of $\frac{e^{-\frac{(x-pe^{-t})^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$

We have seen that the $\frac{e^{-\frac{(x-pe^{-t})^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ fundamental function satisfies both the forward and backward equations, therefore its Laplace transform must satisfy both of the ordinary differential equations:

$$U'' + 2pU' + 2(1-s)U = 0 \tag{3.1}$$

$$U'' - 2pU' - 2sU = 0. \tag{3.2}$$

Because of (3.1) must be of the form: $HF_1 + KxF_2$, where H and K must be some linear combinations of G_1 and pG_2 since it satisfies (3.2) as well. Let us observe that $\frac{e^{-\frac{(x-pe^{-t})^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ is analytic in x for all values p and t except when $t = 0$ and $x = p$, in the latter case it is undefined. Therefore its Laplace transform is analytic in the $x \le p$ domain as well. Putting $x = 0$ in $\frac{e^{-\frac{(x-pe^{-t})^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ gives $\frac{e^{-\frac{p^2e^{-t2}}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ and we have seen that its Laplace transform is $AG_1 - pG_2$, so $H = AG_1 - pG_2$ (when $x \le p$). The determination of K is more involved. Differentiating the fundamental function by x gives:

$$\frac{2pe^{-t}e^{-\frac{p^2e^{-t2}}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)^{\frac{3}{2}}} = e^{p^2}\,\frac{2pe^{-t}e^{-\frac{p^2e^{-t2}}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)^{\frac{3}{2}}}. \tag{3.3}$$

Clearly the coefficient K is the Laplace transform of (3.3). To evaluate it let us compute the following convolution integral:

$$e^{p^2}\frac{2pe^{-t}e^{-\frac{p^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)^{\frac{3}{2}}} * \frac{1}{\sqrt{\pi}\left(1-e^{-2t}\right)}. \tag{3.4}$$

It has been shown that the Laplace transform of the second factor in (3.4) is A, so the Laplace transform of (3.3) is the Laplace transform of (3.4) divided into A. Next we evaluate (3.4):

$$\int_0^t e^{p^2}\frac{2pe^{-r}e^{-\frac{p^2}{\left(1-e^{-2r}\right)}}}{\sqrt{\pi}\left(1-e^{-2r}\right)^{\frac{3}{2}}}\frac{1}{\sqrt{\pi}\left(1-e^{-2(t-r)}\right)}dr =$$

putting r for e^{-r} yields:

$$\int_T^1 e^{p^2}\frac{2pre^{-\frac{p^2}{\left(1-r^2\right)}}}{\sqrt{\pi}\left(1-r^2\right)^{\frac{3}{2}}}\frac{1}{\sqrt{\pi}\left(r^2-T^2\right)}dr =$$

where $T=e^{-r}$. Substituting $\sqrt{r}$ for r gives:

$$e^{p^2}\int_{T^2}^1 \frac{pe^{-\frac{p^2}{(1-r)}}}{\sqrt{\pi}(1-r)^{\frac{3}{2}}}\frac{1}{\sqrt{\pi}\left(r^2-T^2\right)}dr$$

$$=e^{p^2}\int_0^{1-T^2}\frac{pe^{-\frac{p^2}{\left(1-T^2-r\right)}}}{\sqrt{\pi}\left(1-T^2-r\right)^{\frac{3}{2}}}\frac{1}{\sqrt{\pi r}}dr$$

$$=e^{-p^2}\cdot\frac{pe^{-\frac{p^2}{t}}}{\sqrt{\pi t^3}} * \frac{1}{\sqrt{\pi t}}\bigg|_{t=1-T^2}=\frac{e^{-\frac{p^2e-2t}{(1-2^{-2t})}}}{\sqrt{\pi(1-2^{-2t})}}.$$

Thus we have for the coefficient $K=\dfrac{AG_1-pG_2}{A}$. Hence the Laplace transform of $\dfrac{e^{-\frac{(x-pe^{-t})^2}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi}\left(1-e^{-2t}\right)}$ is for $x\le p$:

$$(AG_1 - pG_2)F_1 + xF_2 \frac{AG_1 - pG_2}{A} = \frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A}.$$

Next let us consider the case $p \le x$. If the same computation is repeated but instead of $x = 0$ we look at $p = 0$, that is we compute the coefficients of G_1 and pG_2. Putting $p = 0$ in $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}$ gives $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}$. Its Laplace transform is $AF_1 - xF_2$, carrying through similar computation as was done for the coefficient of xF_2 we get for the coefficient for $pG_2 \frac{AF_1 - xF_2}{A}$. Thus the Laplace transform of $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}$ when $p \le x$ is: $\frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A}$. Hence the Laplace transform of $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}$ is:

$$?\left(\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}\right) = \begin{cases} \frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A} & \text{if } p \le x \\ \frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A} & \text{if } x \le p. \end{cases} \tag{3.5}$$

4 Level Crossing Probabilites

Let the random variable FC or $FC(x)$ (first crossing) be the smallest possible value of t such that $X_t = x$ given $X_0 = p$. Let $\varphi(t, p, x)$ be the distribution of FC, clearly: $\varphi(t, p, x) * P(t, x, x) = P(t, p, x)$.

That is: $\varphi(t, p, x) * \frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}} = \frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}$.

Now the probability that X_t stays below x is: $P\left(\sup_{0\le r\le t} X_r\right) = 1 - \int_0^t \varphi(r)dr$.

Let us denote the Laplace transform of φ by Ψ, then Ψ for $0 \le p \le x$ using (3.5) can be expressed as

$$\psi = \frac{(AG_1(p) + pG_2(p))(AF_1(x) + xF_2(x))}{(AG_1(x) + xG_2(x))(AF_1(x) + xF_2(x))} \tag{4.1}$$

$$= \frac{AG_1 + pG_2}{AG_1 + xG_2} \tag{4.2}$$

For the special case when $p = 0$, that is when X_t reaches level x subject to the initial condition $X_0 = 0$ is

$$\psi = \frac{A}{AG_1 + xG_2}. \tag{4.3}$$

For this case Bellman and Harris $[1]$ found the following expression:

$$\psi = \frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{\int_0^\infty e^{-y^{2+2xy}y^{s-1}dy}}. \tag{4.4}$$

For the case $p \ge x \ge 0$:

$$\psi(p,x) = \frac{\frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A}}{\frac{(AF_1 + xF_2)(AG_1 - xG_2)}{A}} \tag{4.5}$$

$$= \frac{AG_1 - pG_2}{AG_1 - xG_2} \tag{4.6}$$

For the special case $p > 0$, $x = 0$ we have:

$$\psi(p,0) = \frac{AG_1 - pG_2}{A}. \tag{4.7}$$

Using (4.7) it is not difficult to show that (4.2) holds for $p \le x$ and holds for $p \ge x$ as well. Formula (4.7) easily invertable, for

$$\frac{1}{A}=\frac{2\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}=\frac{s\Gamma\left(\frac{s+1}{2}\right)}{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)}=\frac{2\Gamma\left(\frac{s+1}{2}\right)}{\frac{s}{2}\Gamma\left(\frac{s}{2}+1\right)}.$$

Clearly $\dfrac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}$ is the Laplace transform of $2\cdot\dfrac{e^{-t}}{\sqrt{\pi\left(1-e^{-2t}\right)}}$. Hence (4.7) is the Laplace transform of:

$$2\cdot\frac{d}{dt}\left\{\frac{e^{-\frac{p^2e^{-2t}}{\left(1-e^{-2t}\right)}}}{\sqrt{\pi\left(1-e^{-2t}\right)}}*\frac{e^{-t}}{\sqrt{\pi\left(1-e^{-2t}\right)}}\right\}=2\cdot\frac{d}{dt}\frac{1}{\sqrt{\pi}}\int_{\frac{pe^{-t}}{\sqrt{1-e^{-2t}}}}^{\infty}e^{-z^2}dz=\frac{2pe^{-t}}{\sqrt{\pi}}\frac{e^{-\frac{p^2e^{-2t}}{1-e^{-2t}}}}{\left(1-e^{-2t}\right)^{\frac{3}{2}}}.$$

5 The Equivalence of Bellman-Haris' and our Result

To show that formulas (4.3) and (4.4) are the same, we have to evaluate the integral $\int_0^{\infty}e^{-y^2+2xy}=e^{x^2}\cdot e^{-(x-y)^2}=e^{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}\frac{d^ne^{-y^2}}{dy^n}$.

Substituting this into the integral we get:

$$\int_0^{\infty}e^{-y^2+2xy}y^{s-1}dy=e^{x^2}\cdot\int_0^{\infty}e^{(x-y)^2}y^{s-1}dy=e^{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}\int_0^{\infty}\frac{d^ne^{-y^2}}{dy^n}y^{s-1}dy. \tag{5.1}$$

Let us observe that the integrals on the right hand side are the Mellin transfroms of the functions $\dfrac{d^ne^{-y^2}}{dy^n}$. First we compute the Mellin transform of e^{-y^2} which is:

$$\int_0^{\infty}e^{-y^2}y^{s-1}dy=\frac{1}{2}\int_0^{\infty}e^{-y}y^{\frac{s}{2}-1}dy=\frac{1}{2}\Gamma\left(\frac{s}{2}\right).$$

Let us denote the Mellin transform of a function f by ? or F. It is not difficult to see that:

$$M(f') = -(s-1)F(s-1)$$
$$M(f'') = -(s-1)(s-2)F(s-2)$$
$$\cdots$$
$$\cdots$$
$$M(f^n) = (-1)^n (s-1)(s-2)\cdots(s-n)F(s-n).$$

Hence the Mellin transforms of $e^{-y^2}, \frac{de^{-y^2}}{dy}, \frac{d^2e^{-y^2}}{dy^2}, \frac{d^3e^{-y^2}}{dy^3}\ldots$ are

$$\frac{1}{2}\Gamma\left(\frac{s}{2}\right), -\frac{(s-1)}{2}\Gamma\frac{(s-1)}{2}, \frac{(s-2)(s-1)}{2}\Gamma\frac{(S-2)}{2}, \frac{(s-3)(s-2)(s-1)}{2}\Gamma\frac{(S-3)}{2},$$
$$\frac{(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\frac{(S-4)}{2}, \frac{(s-5)(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\frac{(S-5)}{2},\cdots.$$

Substituing these into (5.1) gives:

$$e^{x^2}\left\{\frac{1}{2}\Gamma\left(\frac{s}{2}\right) + \frac{x}{1!}\frac{s-1}{2}\Gamma\frac{(s-1)}{2} + \frac{x^2}{2!}\frac{(s-2)(s-1)}{2}\Gamma\frac{(s-2)}{2}\right.$$
$$+\frac{x^3}{3!}\frac{(s-3)(s-2)(s-1)}{2}\Gamma\frac{(s-3)}{2} + \frac{x^4}{4!}\frac{(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\frac{(s-4)}{2}$$
$$\left.+\frac{x^5}{5!}\frac{(s-5)(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\frac{(s-5)}{2} + \cdots\right\}$$
$$= e^{x^2}\left\{\frac{1}{2}\Gamma\left(\frac{s}{2}\right) + \frac{x^2}{2!}\frac{(s-2)(s-1)}{2}\Gamma\frac{(s-2)}{2}\right.$$
$$\left.+\frac{x^4}{4!}\frac{(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\frac{(s-4)}{2} + \cdots\right\}$$
$$+e^{x^2}\left\{\frac{x}{1!}\frac{s-1}{2}\Gamma\frac{(s-1)}{2} + \frac{x^3}{3!}\frac{(s-3)(s-2)(s-1)}{2}\Gamma\frac{(s-3)}{2}\right.$$
$$\left.+\frac{x^5}{5!}\frac{(s-5)(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\frac{(s-5)}{2} + \cdots\right\}$$
$$= e^{x^2}\left\{\frac{1}{2}\Gamma\left(\frac{s}{2}\right) + \frac{x^2}{2!}2^0\Gamma(s-1)\Gamma\left(\frac{s}{2}\right) + \frac{x^4}{4!}2^1(s-3)(s-1)\Gamma\left(\frac{s}{2}\right) + \cdots\right\}$$

$$+e^{x^2}\left\{\frac{x}{1!}2^0\Gamma\left(\frac{s+1}{2}\right)+\frac{x^3}{3!}2^1(s-2)\Gamma\left(\frac{s+1}{2}\right)\right.$$

$$\left.+\frac{x^5}{5!}2^2(s-4)(s-2)(s-1)\Gamma\left(\frac{s+1}{2}\right)+\cdots\right\}$$

$$=e^{x^2}\frac{1}{2}\Gamma\left(\frac{s}{2}\right)\left\{1-\frac{x^2}{1!}\frac{\frac{1-s}{2}}{\frac{1}{2}}+\frac{x^4}{2!}\frac{\frac{1-s}{2}}{\frac{1}{2}}\frac{\frac{3-s}{2}}{\frac{3}{2}}+\cdots\right\}$$

$$+e^{x^2}\Gamma\left(\frac{s+1}{2}\right)x\left\{1-\frac{x^2}{1!}\frac{1-\frac{s}{2}}{\frac{3}{2}}+\frac{x^4}{2!}\frac{1-\frac{s}{2}}{\frac{3}{2}}\frac{2-\frac{s}{2}}{\frac{5}{2}}+\cdots\right\}$$

$$=e^{x^2}\frac{1}{2}\Gamma\left(\frac{s}{2}\right)F\left(\frac{1-s}{2};\frac{1}{2},-x^2\right)+e^{x^2}\Gamma\left(\frac{s+1}{2}\right)xF\left(1-\frac{s}{2},\frac{3}{2};-x^2\right)$$

$$=\frac{1}{2}\Gamma\left(\frac{s}{2}\right)F\left(\frac{s}{2},\frac{1}{2};-x^2\right)+\Gamma\left(\frac{s+1}{2}\right)xF\left(\frac{1+s}{2},\frac{3}{2};x^2\right).$$

Hence Bellman and Harrises formula becomes:

$$\frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{\int_0^\infty e^{-y^2+2xy}y^{s-1}dy}=\frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)F\left(\frac{s}{2},\frac{1}{2};x^2\right)+\Gamma\left(\frac{s+1}{2}\right)xF\left(\frac{1+s}{2},\frac{3}{2};x^2\right)}.$$

Diving both the numerator and the denominator of the right hand side into $\Gamma\left(\frac{s+1}{2}\right)$ gives $\frac{A}{AG_1+xG_2}$ and this completes the proof.

Reference

[1] Bellman, R., Harris, T.: Recurence times for the Ehrenfest model. Pacific J. Math. 1. 179-193 (1951)