Analytical Upper Bound for the Error on the Discretization of Uncertain Linear Systems by using the Tensor Product Model Transformation

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Abstract: This work provides analytical upper bounds on the discretization error of uncertain linear systems. The Tensor Product Model Transformation is used to approximate the derived discretized system, with a reduced number of vertices. Digital state feedback controllers are then designed for the discretized system, for comparison to other available work in the current literature.

Keywords: Tensor Product Model Transformation; Discretization; Uncertain Linear Systems

1   Introduction

The technological development of high-performance computers and microprocessors allows for the usage of digital controllers [1], [2]. In real world applications, digital controllers have been widely used to control dynamical continuous-time systems, mainly due to their multiply advantages, such as, lower power consumption and fabrication costs, improved flexibility, ease in reprogramming the same device to deal with different control strategies and also implementation of complex digital control laws, the possibility of developing of interfaces with users (including web interfaces) and last, but not least, greater reliability [1], [3].

There are three main techniques used to synthesize digital controllers [4], [5], [6]: (i) the emulation design approach, where a continuous-time controller is designed regardless of the sampling time and, then, the controller is sampled [7], [8]; (ii) the discrete-time approach, where the design is done from a discrete-time description of the process which represents its behavior only in the sampling times, which means that the intersampling behavior are neglected [9], [10]; and, finally, (iii) the sampled-data design, where the acquired controller is synthesized based on a discrete-time model that takes into account the system behavior at the sampling and also in the intersampling times [11], [12].

The discretized version of the continuous-time system can be obtained by employing, for instance, the Taylor series expansion to deal with the exponential of the system dynamic matrix. Nevertheless, such technique can be directly applied only for systems without uncertainties. However, it is well known that real systems are usually affected by uncertainties which denote, for instance, neglected dynamics, external disturbances, unknown parameters, noise associated with the collected information or measurements, or the inaccuracy of sensors and actuators [13], making the discretization procedure of an uncertain system, which requires to compute the exponential of an uncertain matrix, a hard problem to deal with. To overcome such challenge, several numerical strategies have been used, such as Chebyshev quadrature formula and internal arithmetic, Jordan decomposition, or the Cayley-Hamilton theorem [14], [15], or, as more frequently found, a first-order Taylor series expansion technique [16], [17], [18]. All of these methodologies can be employed only for systems with a small number of vertices and, especially, the latter, yields an inaccurate discrete-time model, mainly for larger sampling times.

A more recent result [9], uses a technique based on a Taylor series expansion of a fixed order to obtain the discrete-time representation of the continuous-time system, whose discrete-time model is composed of homogeneous polynomial matrices plus an additive norm-bounded term that represents the discretization residual error. Albeit, such procedure produces a more precise description of the systems dynamics, as a drawback, the discrete-time representation depends on multiple indexes increasing the number of Linear Matrix Inequalities (LMIs) to be solved, in the synthesis conditions, as the chosen order augments. In order to avoid the aforementioned problem, the work in [19] proposed an approach based on a grid of the possible values for the matrix exponential function and an application of the tensor product model transformation technique to acquire a suitable polytopic model, reducing the number of LMIs to be solved. However, the error committed by the discretization technique is ignored.

The Tensor Product Model Transformation (TPMT) [20], [21], [22], [23], [24], [25] is a numerical technique that allows one to extract a convex representation, similar to a Takagi-Sugeno (TS) representation for a quasi-Linear Parameter Varying representation of a dynamical system. It makes use of the Higher-Order Singular Value Decomposition (HOSVD) to numerically extract a meaningful

representation from a sampled representation of a function over a grid of possible values. This convex representation allows the use of readily available Linear Matrix Inequality (LMI) conditions to be used for the synthesis of nonlinear systems in a systematic approach [24].

Several works in the literature focused on different applications of the TPMT to different control problems, and of special note is the work in [19], in which the authors made use of the Tensor Product Model Transformation to find a convex representation for the discretization of Uncertain Linear Systems. However, their work does not take into account the error made on this discretization procedure by using a grid of possible values for the discrete-time representation of the uncertain system. In that regard, this work extends the ideas introduced in [19], providing an analytical upper bound for the residual error norm and a discrete-time polytopic model for the continuous-time uncertain system.

Notation

In this paper, lowercase variables represent scalars, lowercase boldface variables represent column vectors, uppercase variables represent matrices and calligraphic uppercase variables represent tensors. S^T denotes the transpose of matrix S, $Q > 0$ ($Q \geq 0$) indicates that matrix Q is positive definite (positive semi-definite), and $\star$ indicates terms that can be inferred from symmetry on a symmetric matrix. $\mathcal{L} \times_n U$ represents the n-mode product between tensor $\mathcal{L}$ and matrix U. In order to get acquainted with the multilinear algebra operations used in this paper, we refer the reader to [20], [27].

2   Discretization Strategy

Consider the uncertain linear system described by the polytopic model

$$\dot{x} = A(\boldsymbol{\alpha})x + B(\boldsymbol{\alpha})\boldsymbol{u} = \sum_{i=1}^{r} \alpha_i(A_i x + B_i \boldsymbol{u}) \tag{1}$$

with $\boldsymbol{x} \in \mathbb{R}^n$ the system's states, $\boldsymbol{u} \in \mathbb{R}^m$ the control inputs, $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$ the system matrices and α_i the uncertain convex weights of the model, such that:

$$\alpha_i \in [0,1], \ \sum_{i=1}^{r} \alpha_i = 1. \tag{2}$$

In this paper, our aim is to find an approximate uncertain discrete time polytopic model, for this system, described by:

$$\boldsymbol{x_{k+1}} = \left(\hat{A}(\boldsymbol{\beta}) + \Delta A\right)\boldsymbol{x_k} + \left(\hat{B}(\boldsymbol{\beta}) + \Delta B\right)\boldsymbol{u_k}$$
$$= \sum_{i=1}^{\hat{r}} \beta_i\left((\hat{A}_i + \Delta A)\boldsymbol{x_k} + (\hat{B}_i + \Delta B)\boldsymbol{u_k}\right) \tag{3}$$

with $\hat{r}$ the number of linear models composing the uncertain discretized system, β_i the unknown convex weights with:

$\beta_i \in [0,1]$, $\sum_{i=1}^{\hat{r}} \beta_i = 1$ $\hspace{6cm}$ (4)

and ΔA and ΔB norm bounded uncertainties with:

$||\Delta A||_2 \le \eta_a$, $||\Delta B||_2 \le \eta_b$. $\hspace{6cm}$ (5)

By considering that the uncertain convex weights α_i are constant over time and that the control inputs are constant over the sampling period, the system described in (1) can be exactly discretized by:

$$\boldsymbol{x_{k+1}} = e^{A(\boldsymbol{\alpha})\tau}\boldsymbol{x_k} + \int_0^\tau \left(e^{A(\boldsymbol{\alpha})t} dt\right) B(\boldsymbol{\alpha})\boldsymbol{u_k} \hspace{3cm} (6)$$

with τ the sampling period. Similarly to [19], our problem can be restated as finding a convex representation for:

$$\hat{A}(\boldsymbol{\alpha}) = e^{A(\boldsymbol{\alpha})\tau}, \ \hat{B}(\boldsymbol{\alpha}) = \int_0^\tau \left(e^{A(\boldsymbol{\alpha})t} dt\right) B(\boldsymbol{\alpha}), \hspace{2.5cm} (7)$$

and the Tensor Product Model Transformation (TPMT) [20], [21], [22], [23] can be employed to this end. Unlike [19] though, we explicitly consider the discretization error introduced by the sampling step of the TPMT, which are represented by ΔA and ΔB in (3). The main contribution in this paper can then be understood as analytical upper bounds on η_a and η_b.

2.1 Tensor Product Model Transformation (TPMT)

In order to find a representation with the smallest $\hat{r}$ (or the smallest number of linear systems composing the uncertain model), we define the matrix function:

$$H(\boldsymbol{\alpha}) = [\hat{A}(\boldsymbol{\alpha}) \quad \hat{B}(\boldsymbol{\alpha})] \hspace{5cm} (8)$$

which will be approximated by the Tensor Product Model Transformation. This approach is common in the TPMT literature and allows for a smaller number of vertices found in the end, when compared against approximating the matrix functions separately and joining the convex models found afterwards.

The TPMT can usually be divided into four steps: sampling, Higher Order Singular Value Decomposition (HOSVD), Convex Hull Manipulation and Interpolation [26].

2.1.1 Sampling

If done on the usual approach, the sampling step would be performed by defining a regular sampling grid over the hyperrectangular domain $\boldsymbol{\alpha} \in [0,1]^r$ and, if we considered sampling each α_i with p samples, storing it on a tensor $\mathcal{H} \in \mathbb{R}^{p \times \dots \times p \times n \times (n+m)}$.

This approach is usually employed with Linear Parameter Varying (LPV) and Takagi-Sugeno (TS) models, since it allows for a special structure in which the resulting weights can be decomposed as functions of a single scalar variable.

When such structure on the weighting functions is not needed, an approach similar to the 1-level Nested Tensor Product Model Transformation (NTPMT) [21] could be used instead, resulting in a tighter model with a smaller number of linear systems composing the desired model.

We make use, instead, of the approach proposed in [19] since we have a particular structure on our domain. From (2) we know that:

$$\alpha_r = 1 - \sum_{i=1}^{r-1} \alpha_i \tag{9}$$

and we need only to sample the first $r - 1$ dimensions of $\boldsymbol{\alpha}$. In addition to this, we only consider *valid samples* as being those for which:

$$\sum_{i=1}^{r-1} \alpha_i \leq 1. \tag{10}$$

Inspired by the 1-level NPTMT, we store the valid samples on a tensor $\mathcal{H} \in \mathbb{R}^{\kappa \times n \times (n+m)}$, with κ the number of *valid samples* taken and $\mathcal{H}_{ijk}$ the element in row j and column k from sample i.

With this sampling, the matrix function can be represented as:

$$H(\boldsymbol{\alpha}) = \mathcal{H} \times_1 \boldsymbol{w}^T(\boldsymbol{\alpha}) + \Delta H \tag{11}$$

with $\boldsymbol{w}^T(\boldsymbol{\alpha})$ an interpolation function that assigns a weight to each sample depending on the value of $\boldsymbol{\alpha}$ and ΔH the interpolation/grid sampling error. While any interpolation strategy that yields convex weights could be used, like a finite element interpolation for instance, in this work we consider a nearest neighbor, or piecewise constant, interpolation to derive an upper bound on the norm of the part that compose ΔH (ΔA e ΔB).

2.1.2   Higher Order Singular Value Decomposition (HOSVD)

By making use of the HOSVD [27] along the first direction of $\mathcal{H}$, it can be rewritten as:

$$\mathcal{H} = \mathcal{L} \times_1 U_1 \tag{12}$$

with $\mathcal{L} \in \mathbb{R}^{q \times n \times (n+m)}$, $U_1 \in \mathbb{R}^{\kappa \times q}$, and the equality is ensured in the equation above only on the cases in which no nonzero higher order singular values are discarded. In case nonzero higher order singular values are discarded, the approach presented in [26] can be used to find an extra uncertainty that needs to be considered in ΔH.

2.1.3   Convex Hull Manipulation

In order for us to retrieve an interesting convex representation for function $H(\alpha)$, it is necessary to impose some special properties upon matrix U_1. The following properties are common in the TPMT literature:

- Sum Normalization (SN): for every row of U_1, the sum of its columns is equal to one

- Non Negative (NN): every element of U_1 is nonnegative

- Inverse Normalized (INO): the minimum of every column of U_1 is the same and is equal to zero

- Relaxed Normalized (RNO): the maximum of every column of U_1 is the same

- Close to Normalized (CNO): the simplex formed by the unitary vectors is the smallest volume simplex that covers the vectors formed by the rows of U_1

The SN and NN properties are the bare minimum to ensure that we retrieve a convex representation, but usually do not generate interesting models by themselves. The other properties are usually employed to guarantee that a "tight" representation is found, meaning that, in this case, they aim at representing the uncertain system with a small set.

In the examples presented later in this work, we make use of the SN-NN transformation [22], followed by the RNO-INO transformation [28] and the CNO transformation.

2.1.4   Interpolation

If the combined transformations are such that:

$$\hat{U}_1 = U_1 T_1 \Rightarrow U_1 = \hat{U}_1 T_1^{-1} \tag{13}$$

with $\hat{U}_1$ the matrix with the desired properties and T_1 the nonsingular matrix that transforms the original matrix. Then by combining (11), (12) and (13) we get that

$$
\begin{aligned}
H(\boldsymbol{\alpha}) &= (\mathcal{L} \times_1 U_1) \times_1 \boldsymbol{w}^T(\boldsymbol{\alpha}) + \Delta H \\
&= \left(\mathcal{L} \times_1 \hat{U}_1 T_1^{-1}\right) \times_1 \boldsymbol{w}^T(\boldsymbol{\alpha}) + \Delta H \\
&= (\mathcal{L} \times_1 T_1^{-1}) \times_1 \boldsymbol{w}^T(\boldsymbol{\alpha}) \hat{U}_1 + \Delta H \\
&= \hat{\mathcal{L}} \times_1 \hat{\boldsymbol{w}}^T(\boldsymbol{\alpha}) + \Delta H \\
&= \sum_{i=1}^{\hat{r}} \beta_i \begin{bmatrix} \hat{A}_i & \hat{B}_i \end{bmatrix} + \Delta H
\end{aligned}
\tag{14}
$$

with $\beta_i = w_i(\boldsymbol{\alpha})$, $\hat{A}_{i_{jk}} = \hat{\mathcal{L}}_{ijk}$ and $\hat{B}_{i_{jk}} = \hat{\mathcal{L}}_{ij(k+n)}$. Note that, even though we present a form to calculate β_I since the values of $\boldsymbol{\alpha}$ are unknown, and we consider the values of β_I to be unkown as well, they need not be determined.

2.2   Analytical Upper Bound on the Grid Sampling Step

By comparing (14) and (3), we get that the error in (14) can be rewritten as:

$$\Delta H = \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \tag{15}$$

Note that, since we are using a nearest neighbour interpolation these errors can be taken as the error in this kind of interpolation on the sampling grid. As such, consider that $\boldsymbol{\alpha}$ represents any point on the domain, while $\boldsymbol{\alpha_g}$ represents the nearest point on the grid. We get that the interpolation errors can be written as:

$$\Delta A = e^{A(\boldsymbol{\alpha})\tau} - e^{A(\boldsymbol{\alpha_g})\tau} \tag{16}$$

$$\Delta B = \int_0^\tau \left(e^{A(\boldsymbol{\alpha})t}dt\right) B(\boldsymbol{\alpha}) - \int_0^\tau \left(e^{A(\boldsymbol{\alpha_g})t}dt\right) B\big(\boldsymbol{\alpha_g}\big) \tag{17}$$

Since $\boldsymbol{\alpha_g}$ represents the nearest point on the grid, (16) can be rewritten as:

$$\Delta A = e^{A(\boldsymbol{\alpha_g})\tau + A(\boldsymbol{\delta_\alpha})\tau} - e^{A(\boldsymbol{\alpha_g})\tau} \tag{18}$$

with

$$A(\boldsymbol{\delta_\alpha}) = \sum_{i=1}^r \left(\alpha_i - \alpha_{g_i}\right) A_i \tag{19}$$

From the definition of the matrix exponential with its Taylor series, we get that:

$$\Delta A = \sum_{i=0}^\infty \left(\frac{\left(A(\boldsymbol{\alpha_g})\tau + A(\boldsymbol{\delta_\alpha})\tau\right)^i}{i!} - \frac{\left(A(\boldsymbol{\alpha_g})\tau\right)^i}{i!}\right)$$

$$= \sum_{i=0}^\infty \frac{1}{i!} \left(\sum_{k=0}^i \binom{i}{k} \left(A(\boldsymbol{\alpha_g})\tau\right)^k (A(\boldsymbol{\delta_\alpha})\tau)^{i-k} - \left(A(\boldsymbol{\alpha_g})\tau\right)^i\right)$$

$$= \sum_{i=0}^\infty \frac{1}{i!} \left(\sum_{k=0}^{i-1} \binom{i}{k} \left(A(\boldsymbol{\alpha_g})\tau\right)^k (A(\boldsymbol{\delta_\alpha})\tau)^{i-k}\right) \tag{20}$$

By taking the norm on both sides, we can write that:

$$||\Delta A||_2 \le \eta_a = \sum_{i=0}^\infty \sum_{k=0}^{i-1} \frac{1}{i!} \binom{i}{k} ||A(\boldsymbol{\alpha_g})||_2^k ||A(\delta_\alpha)||_2^{i-k} \tau^i \tag{21}$$

Once again, since $\boldsymbol{\alpha_g}$ represents the nearest point on the grid, (17) can be rewritten as:

$$\Delta B = \int_0^\tau \left(e^{A(\boldsymbol{\alpha})t} - e^{A(\boldsymbol{\alpha_g})t}dt\right) B(\boldsymbol{\alpha}) + \int_0^\tau \left(e^{A(\boldsymbol{\alpha_g})t}dt\right) \left(B(\boldsymbol{\alpha_g}) - B(\boldsymbol{\alpha})\right) \tag{22}$$

By taking the norm on both sides of (22) and making use of the triangular inequality, we get that:

$$||\Delta B||_2 \le || \int_0^\tau \left(e^{A(\boldsymbol{\alpha})t} - e^{A(\boldsymbol{\alpha_g})t}dt\right) B(\boldsymbol{\alpha})||_2$$

$$+ || \int_0^\tau \left(e^{A(\boldsymbol{\alpha_g})t}dt\right) \left(B(\boldsymbol{\delta_\alpha})\right)||_2 \tag{23}$$

with

$$B(\boldsymbol{\delta_\alpha}) = \sum_{i=1}^r \left(\alpha_i - \alpha_{g_i}\right) B_i \tag{24}$$

The first term on the right hand side of (23) can be upper bounded, by making use of the developments in (20), by:

$$|| \int_0^\tau \left(e^{A(\alpha)t} - e^{A(\alpha_g)t} dt \right) B(\alpha)||_2 \leq$$
$$\int_0^\tau \sum_{i=0}^\infty \frac{t^i}{i!} \left(\sum_{k=0}^{i-1} \binom{i}{k} ||A(\alpha_g)||_2^k ||A(\delta_\alpha)||_2^{i-k} \right) dt \, ||B(\alpha)||_2 \leq$$
$$\sum_{i=0}^\infty \frac{t^{i+1}}{(i+1)!} \left(\sum_{k=0}^{i-1} \binom{i}{k} ||A(\alpha_g)||_2^k ||A(\delta_\alpha)||_2^{i-k} \right) ||B(\alpha)||_2 \tag{25}$$

whereas the second term can be upper bounded by:

$$|| \int_0^\tau \left(e^{A(\alpha_g)t} dt \right) \left(B(\delta_\alpha) \right) ||_2 \leq \int_0^\tau ||e^{A(\alpha_g)t}||_2 ||B(\delta_\alpha)||_2 dt \leq$$
$$\int_0^\tau e^{||A(\alpha_g)||_2 t} ||B(\delta_\alpha)||_2 dt = \frac{||B(\delta_\alpha)||_2}{||A(\alpha_g)||_2} \left(e^{||A(\alpha_g)||_2 \tau} - 1 \right) \tag{26}$$

By considering (25) and (26), we can write that:

$$||\Delta B||_2 \leq \eta_b \tag{27}$$

with

$$\eta_b = \sum_{i=0}^\infty \frac{t^{i+1}}{(i+1)!} \left(\sum_{k=0}^{i-1} \binom{i}{k} ||A(\alpha_g)||_2^k ||A(\delta_\alpha)||_2^{i-k} \right) ||B(\alpha)||_2 +$$
$$\int_0^\tau e^{||A(\alpha_g)||_2 t} ||B(\delta_\alpha)||_2 dt = \frac{||B(\delta_\alpha)||_2}{||A(\alpha_g)||_2} \left(e^{||A(\alpha_g)||_2 \tau} - 1 \right) \tag{28}$$

Equations (21) and (28) allow us to find an upper bound on the grid sampling step and can be used to determine bounds for the norm-bounded uncertainty in (3). In order to make use of these equations, though, we need to be able to calculate $||A(\alpha_g)||_2$, $||B(\alpha_g)||_2$, $||A(\delta_\alpha)||_2$ and $||B(\delta_\alpha)||_2$.

To do so, we consider that the sampling grid is such that:

$$|\alpha_{i_\ell} - \alpha_{g_{i_\ell}}| \leq h \tag{29}$$

By using this limit, together with the fact that:

$$\sum_{i=1}^r \left(\alpha_i - \alpha_{g_i} \right) = 0 \tag{30}$$

and that the intersection of a polytope and a hyperplane is always a polytope, we can write:

$$\alpha - \alpha_g = \sum_{k=1}^{n_v} \gamma_k v_k \tag{31}$$

with v_k the vertices of the intersection from the $[-h, h]^r$ polytope with the zero-sum hyperplane and

$$\gamma_k \in [0,1], \ \sum_{k=1}^{n_v} \gamma_k = 1 \tag{32}$$

Finally, with these definitions, we have that:

$$||A(\alpha_g)||_2 \leq \max_i ||A_i||_2 \tag{33}$$

$$||B(\alpha_g)||_2 \leq \max_i ||B_i||_2 \tag{34}$$

$$||A(\delta_\alpha)||_2 \leq \max_k || \sum_{i=1}^r v_{ki} A_i ||_2 \tag{35}$$

$$||B(\delta_\alpha)||_2 \leq \max_k || \sum_{i=1}^r v_{ki} B_i ||_2 \tag{36}$$

3   Illustrative Examples

To illustrate the advantages of the proposed discretization procedure, we synthesize digital state feedback controllers based on the derived discretized model. For that, we adapted the controller design condition provided in [30], which is given in the following theorem.

Theorem 1 (adapted from [30]). For given η_A and η_B, if there exist positive definite matrices $P_i = P_i^T \in \mathbb{R}^{n \times n}$, $i = 1, \dots, \hat{r}$, matrices $G \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{m \times n}$, and a scalar μ such that:

$$\begin{bmatrix} P_i - G^T - G & \star & \star & \star \\ \hat{A}_i G + \hat{B}_i X & -P_i + \mu(\eta_A^2 + \eta_B^2)I & \star & \star \\ G & 0 & -\mu I & \star \\ X & 0 & 0 & -\mu I \end{bmatrix} < 0, \forall i \tag{37}$$

then there exists a digital state feedback controller given by $K = XG^{-1}$ that asymptotically stabilizes system (1).

We compare our discretization procedure with the ones of [10] and [30] in terms of the maximum sampling period and the bounds of the discretization error. Notice that our approach and the one of [30] provide distinct upper bounds for matrices $\hat{A}(\beta)$ and $\hat{B}(\beta)$, whereas the approach of [10] provides a single upper bound for both matrices. Two numerical examples are provided to this end. It is important to emphasize that all comparisons are performed among methods that provide a theoretical bound for the system's discretization error.

Example 1. Consider a linearized inverted pendulum on cart model given by equation (1) with:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{b(\alpha_1)}{M(\alpha_2)} & -\frac{mg}{M(\alpha_2)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b(\alpha_1)}{M(\alpha_2)l} & \frac{(M(\alpha_2)+m)g}{M(\alpha_2)l} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{M(\alpha_2)} \\ 0 \\ -\frac{1}{M(\alpha_2)l} \end{bmatrix} \tag{38}$$

where $g = 9.8$m/s^2 is the gravity, $l = 0.4$m is the length of the rod, $m = 0.11$kg is the mass of the pendulum (assumed concentrated at the end of the rod), $b(\alpha_1) \in [0.0475\ 0.0525]$Ns/m is the friction coefficient of the cart, and $M(\alpha_2) \in [0.896\ 1.344]$kg is the mass of the cart.

Our aim in this example is determine the maximum discretization period that ensures the closed loop system (with a digital controller) is asymptotically stable.

Applying the conditions of Theorem 1 with 111 points and a tolerance of 10^{-6} provided a maximum period of $\tau = 178$ms with error bounds for the discretization procedure of $\eta_A = 0.0043$ and $\eta_B = 0.0012$. Contrast this sampling time with the maximum ones obtained by the methods of [30], $\tau = 103$ms, and of [10], $\tau = 178$ms, with error bounds of $\eta_A = 0.0459$ and $\eta_B = 0.0051$, and $\eta_A = \eta_B = 0.0473$, respectively. For the method of [30], a truncated 8th order Taylor series expansion was adopted, polynomial matrices of degree 8 and $L = 200$ (a parameter used by [30] to calculate the discretization error); whereas, for the approach of [10], a truncated 10th order Taylor series expansion is considered with affine-dependent optimization variables. Notice that our discretization period is about 1.7 times greater than the one of [30] whereas it is the same as [10]. Nevertheless, our approach does not require determining the Taylor series expansion of each matrix in order to obtain a discretized model and, furthermore, the analytic upper bound for the discretization error is about 10 times smaller than in the other methods. **Table 1** summarizes the results attained for each aforementioned method.

Table 1

Summary of the discretization procedures results for the system of Example 1

Method	Max. Samp. Time	η_A	η_B
Theorem 1	178 ms	0.0043	0.0012
[30]	103 ms	0.0463	0.0051
[10]	178 ms	0.0473	

To further illustrate the performance of our approach, a time simulation of the continuous system with the digital controller is performed and the attained results are depicted in **Figure 1** and **Figure 2**. Starting from an initial condition of $x_0 = [0 \quad 0 \quad \pi/6 \quad 0]^T$, the simulation was performed for 20 seconds and the controller synthesized for the largest attained sampling time was adopted. Notice that the closed-loop inverted pendulum system stabilizes in about 10 seconds and that the amplitude of the control signal presents reasonable bounds for a real implementation.
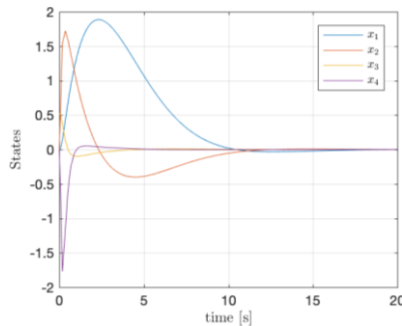

Figure 1

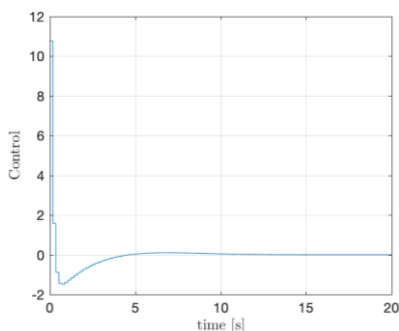Evolution of the inverted pendulum states

Figure 2
Digital control signal applied to the inverted pendulum system

Example 2. In this example an open-loop stable system is considered, which is a simplified two-mass-spring system as proposed in [31] described by equation (1) with the following matrices:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k(\alpha)}{m_1} & \dfrac{k(\alpha)}{m_1} & 0 & 0 \\ \dfrac{k(\alpha)}{m_2} & -\dfrac{k(\alpha)}{m_2} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_1} \\ 0 \end{bmatrix} \tag{39}$$

where $m_1 = m_2 = 1$kg are the masses of carts 1 and 2, respectively, and $k(\alpha) \in [0.5 \ 2.0]$N/m is the spring constant.

The methods of [10] and [30] (using a truncated 10th order Taylor series expansion) are once again applied to the above system and the attained results are compared to our proposed approach. These results are reported in **Table 2**. From the presented data, one can notice that the maximum sampling time attained by our approach is higher than both [10] and [30] and besides the discretization error is about 10 times smaller than the one provided by [10] and 4 times smaller than the one of [30].

Table 2
Maximum sampling times and analytical error bounds for Theorem 1 and the methods of **[10]** and **[30]**

Method	Max. Samp. Time	η_A	η_B
Theorem 1	1.217	0.0266	0.0056
[30]	1.018	0.1909	0.0477
[10]	1.112	0.0987	

Conclusions and Future Works

In this work, we derived an analytical upper bound on the discretization approach proposed in [19]. In order to do so, we utilized a nearest neighbor interpolation in the TPMT and performed manipulations around the definition of the matrix

exponential function. The proposed bounds are considerably smaller than other analytical bounds available for other discretization approaches in the literature.

In the future, we aim to develop tighter bounds for this error, possibly by using different interpolation strategies. Another interesting development would be the use of the NTPMT [21] for the discretization and comparing the results against the learnings in this paper.

Acknowledgement

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