

Implementation of Discrete-Time Fractional-Order Controllers based on LS Approximations

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Abstract: In this paper we develop rational discrete-time approximations (IIR filters) to continuous fractional-order integrators and differentiators of type s^α , $\alpha \in \mathfrak{R}$. For that, it is proposed the adoption of the techniques of Padé, Prony and Shanks usually applied in the signal modelling of deterministic signals. These methods yield suboptimal solutions to the problem which only requires finding the solution of a set of linear equations. The results reveal that this approach gives similar or superior approximations in comparison with other widely used methods. Their effectiveness is illustrated, both in the time and frequency domains, through several examples.

Keywords: IIR filters, rational approximations, digital differentiators, digital integrators, filter design, least-squares, discretization.

1 Introduction

The area of fractional calculus (FC) emerged at the same time as the classical differential calculus and deals with derivatives and integrals to an arbitrary order (real or even complex order) ([1], [2], [3], [4]). However, its inherent complexity postponed the application of the associated concepts. Nowadays, the FC theory is applied in almost all the areas of science and engineering being recognized its ability to yield a superior modelling and control in many dynamical systems ([1], [4], [5], [6], [7], [25]).

In the literature we can find several different definitions for the fractional integration and differentiation of arbitrary order ([1], [2], [4]). One of the most well-known definitions is given by the Grünwald-Letnikov approach ($\alpha \in \mathfrak{R}$):

$${}_a D_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{t-a}{h} \right]} (-1)^k \binom{\alpha}{k} f(t - kh) \quad (1)$$

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)} \quad (2)$$

where $f(t)$ is the applied function, $\Gamma(x)$ is the Gamma function, h is the time increment and $[x]$ means the integer part of x . An important property revealed by equation (1) is that while integer-order operators imply finite series, the fractional-order counterparts are defined by infinite series. This means that integer operators are local operators in opposition with the fractional operators that have, implicitly, a ‘memory’ of all past events.

From a control and signal processing perspective, the Grünwald-Letnikov approach seems to be the most useful and intuitive, particularly for a discrete-time implementation ([4], [8]). Moreover, in the analysis and design of control systems we usually adopt the Laplace transform (L) method. The definition of the fractional-order operator (1) in the Laplace s -domain, under null initial conditions, is given by the relation ($\alpha \in \mathfrak{R}$):

$$L\left\{{}_a D_t^\alpha [f(t)]\right\} = s^\alpha F(s) \quad (3)$$

where $F(s)=L\{f(t)\}$. Note that expression (3) is a direct generalization of the classical integer-order scheme with the multiplication of the signal transform $F(s)$ by the Laplace s -variable raised to a noninteger value α .

Presently, there are several fractional-order control (FOC) strategies where the fractional-order differentiator and/or integrator, s^α ($\alpha \in \mathfrak{R}$), represents its fundamental element. For example, the CRONE¹ controller ([6], [7]) and the fractional PID ($PI^\lambda D^\mu$) controller ([4], [12]) possess a superior performance comparatively with the classical PID controller, particularly when used for the control of fractional-order systems. In general, we may say that the FOC strategies are more flexible and give the possibility of adjusting more carefully the dynamical properties of a control system.

In this paper, we apply the techniques of Padé, Prony and Shanks for obtaining digital rational approximations (IIR filters) to continuous fractional-order integrators and differentiators of type s^α ($\alpha \in \mathfrak{R}$). The resulting approximations are suitable for a digital implementation of a FOC system. The determination process can be synthesized in the following steps:

- 1 Discretize the fractional-order operator s^α using a suitable generating function $H^\alpha(z^{-1})$;
- 2 Obtain the impulse response sequence $h^\alpha(k)$, of the fractional discrete equivalent, by performing a power series expansion (PSE) (or Taylor series) over $H^\alpha(z^{-1})$;

¹ French abbreviation for *Commande Robuste d'Ordre Non Entier*.

- 3 Apply the signal modeling techniques of Padé, Prony or Shanks to $h^\alpha(k)$ in order to get the desired IIR filter approximation.

The proposed method represents an alternative choice to other existing approaches, namely the widely used continued fraction expansion (CFE) method.

Bearing these ideas in mind, the paper is organized as follows. Section 2 presents some discretization schemes for continuous fractional-order integrators and differentiators, while section 3 derives their impulse response sequences. Section 4 develops the signal modeling techniques of Padé, Prony and Shanks for the design of IIR filters approximations to continuous fractional-order operators. Section 5 presents several illustrative examples showing the effectiveness of the new technique. Finally, the main conclusions are drawn.

2 Discretization of Continuous Integrators and Differentiators of Fractional-Order

In general, the discretization of the continuous fractional-order operator s^α ($\alpha \in \mathfrak{R}$) can be expressed by the so-called generating function $s = \omega(z^{-1})$ ([9], [10]). In these $s \rightarrow z$ conversion schemes (also called analog to digital open-loop design methods) the most often used are the Euler (or first backward difference), the Tustin (or bilinear) and the Simpson schemes (see [8]). Recently, new discretization formulae appeared that are weighted interpolations between the Euler-Tustin ([13], [14]) or the Tustin-Simpson ([15], [16]) schemes. For example, the interpolation of 3/4 of the Euler operator with 1/4 of the Tustin operator yields the Al-Alaoui operator (see [14]). This scheme exhibits a much better magnitude fit than the Tustin operator in high frequency range. Table 1 lists the Euler, Tustin and Al-Alaoui operators that will be used in this study.

As can be seen in Table 1, the fractional-order conversion schemes lead to non-rational z -formulae. Therefore, in order to get rational expressions we may get its power series expansion (PSE) (Taylor series) and obtain the final approximation as a truncated z -polynomial function (FIR filter) ([8], [9]). For example, using the Euler operator, $H(z^{-1}) = (1-z^{-1})/T$, and performing a PSE of $[(1-z^{-1})/T]^\alpha$, it yields the discretization formula corresponding to the Grünwald-Letnikov definition (1). Another possible way is to obtain a discrete transfer function in the form of rational function (i.e., as the ratio of two polynomials) (IIR filter) through the application of the continued fraction expansion (CFE) method ([9], [10], [11]).

It is well known that rational approximations frequently converge faster than polynomial approximations and have a wider domain of convergence in the complex domain. Hence, in the work that follows, we develop only IIR-type

approximations of continuous fractional-order integrators and differentiators, which make them suited for z -transform analysis and digital implementation. For that, we propose the use of the least-squares (LS) based methods usually applied in the modeling of deterministic signals (see section 4).

3 Impulse Response of Discretized Integrators and Differentiators of Fractional-Order

This section derives the impulse response sequences $h^\alpha(k)$ of the discretization schemes listed in Table 1. It is assumed that $h^\alpha(k) = 0$ for $k < 0$, corresponding to a causal system.

Expanding the Euler operator $H_E^\alpha(z^{-1})$ into a power series in z^{-1} , we have:

$$\begin{aligned}
H_E^\alpha(z^{-1}) &= \left[\frac{1}{T} (1 - z^{-1}) \right]^\alpha = \\
&= \left(\frac{1}{T} \right)^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^{-k} = \\
&= h_E^\alpha(0) + h_E^\alpha(1)z^{-1} + \dots = \sum_{k=0}^{\infty} h_E^\alpha(k)z^{-k}
\end{aligned} \tag{4}$$

Table 1
Discretization schemes

Method	$s \rightarrow z$ conversion
Euler Grünwald-Letnikov	$s^\alpha \approx \left(\frac{1 - z^{-1}}{T} \right)^\alpha$
Tustin	$s^\alpha \approx \left(\frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \right)^\alpha$
Al-Alaoui	$s^\alpha \approx \left(\frac{8}{7T} \frac{1 - z^{-1}}{1 + z^{-1}/7} \right)^\alpha$

where the impulse response sequence $h_E^\alpha(k)$ is then given by:

$$h_E^\alpha(k) = \left(\frac{1}{T} \right)^\alpha (-1)^k \binom{\alpha}{k}, \quad k \geq 0 \tag{5}$$

By taking the power series expansion (PSE) of the Tustin and Al-Alaoui operators, $H_T^\alpha(z^{-1})$ and $H_A^\alpha(z^{-1})$, respectively, it yields:

$$\begin{aligned} H_T^\alpha(z^{-1}) &= \left(\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} \right)^\alpha = \\ &= \left(\frac{2}{T} \right)^\alpha \sum_{k=0}^{\infty} \left[\sum_{j=0}^k (-1)^j \binom{\alpha}{j} \binom{-\alpha}{k-j} \right] z^{-k} = \\ &= h_T^\alpha(0) + h_T^\alpha(1)z^{-1} + \dots = \sum_{k=0}^{\infty} h_T^\alpha(k)z^{-k} \end{aligned} \quad (6)$$

$$\begin{aligned} H_A^\alpha(z^{-1}) &= \left(\frac{8}{7T} \frac{1-z^{-1}}{1+z^{-1}/7} \right)^\alpha = \\ &= \left(\frac{8}{7T} \right)^\alpha \sum_{k=0}^{\infty} \left[\sum_{j=0}^k (-1)^j \left(\frac{1}{7} \right)^{k-j} \binom{\alpha}{j} \binom{-\alpha}{k-j} \right] z^{-k} = \\ &= h_A^\alpha(0) + h_A^\alpha(1)z^{-1} + \dots = \sum_{k=0}^{\infty} h_A^\alpha(k)z^{-k} \end{aligned} \quad (7)$$

The impulse responses sequences, $h_T^\alpha(k)$ and $h_A^\alpha(k)$, are correspondingly given as ($k \geq 0$):

$$h_T^\alpha(k) = \left(\frac{2}{T} \right)^\alpha \sum_{j=0}^k (-1)^j \binom{\alpha}{j} \binom{-\alpha}{k-j} \quad (8)$$

$$h_A^\alpha(k) = \left(\frac{8}{7T} \right)^\alpha \sum_{j=0}^k (-1)^j \left(\frac{1}{7} \right)^{k-j} \binom{\alpha}{j} \binom{-\alpha}{k-j} \quad (9)$$

Note that the PSE method leads to impulse response sequences of infinite length. For a practically realizable form these sequences must be truncated yielding approximations in the form of FIR (finite impulse response) filters. The $s \rightarrow z$ conversion schemes just described (Euler, Tustin and Al-Alaoui) are special cases of a more general discretization formula called T-integrator (see [21]).

4 Design of IIR Filters based on LS Method

Consider that the impulse response sequence $h^\alpha(k)$ is specified for $k \geq 0$. The IIR filter $H(z^{-1})$, to be designed, has the form:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} = \sum_{k=0}^{\infty} h(k) z^{-k} \quad (10)$$

where $h(k)$ is its impulse response and $m \leq n$.

The IIR filter approximation (10) has $m+n+1$ parameters, namely the coefficients a_k ($k = 1, 2, \dots, n$) and b_k ($k = 0, 1, \dots, m$), which can be selected to minimize some error criterion. Usually, we adopt the least-squares (*LS*) method in order to minimize the error $e_{LS}(k) = h^\alpha(k) - h(k)$, as shown in Figure 1:

$$E_{LS} = \sum_{k=0}^{N-1} [e_{LS}(k)]^2 = \sum_{k=0}^{N-1} [h^\alpha(k) - h(k)]^2 \quad (11)$$

where N denotes the number of samples used in the summation. However, the *LS* approach leads to a nonlinear problem for the model parameters (a_k, b_k), which requires the solution of a set of nonlinear equations and, for that reason, it is often avoided.

If we rewrite (10) as $H(z)A(z) = B(z)$, and assuming that $h^\alpha(k)$ is given approximately by the impulse response of $H(z)$, one can write the corresponding time-domain equation as (note that the left-hand sided corresponds to a convolution):

$$h^\alpha(k) + \sum_{l=1}^n a_l h^\alpha(k-l) = \begin{cases} b_k, & k = 0, 1, \dots, m \\ 0, & k > m \end{cases} \quad (12)$$

This gives a set of linear equations, which can be used in different ways to solve for the coefficients a_k and b_k ([17], [18], [19], [20], [21]). Our objective is to use simple (indirect) methods that can handle more easily the determination of the IIR filter parameters. In this perspective, this study considers the application of three linear suboptimal solutions: the Padé approximation, the Prony's method and the Shanks' method [18].

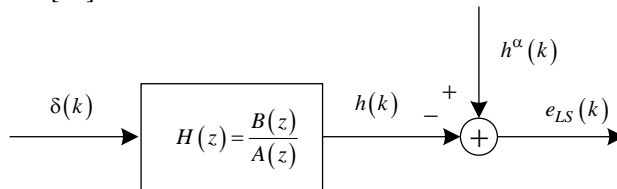


Figure 1
Least-squares method

4.1 Padé Approximation

The Padé approximation yields an IIR filter that fits exactly $h^\alpha(k)$ during the first $m+n+1$ values of k . Then, equation (12) becomes:

$$h^\alpha(k) + \sum_{l=1}^n a_l h^\alpha(k-l) = \begin{cases} b_k, & k = 0, 1, \dots, m \\ 0, & k = m+1, \dots, m+n \end{cases} \quad (13)$$

where $h^\alpha(k) = 0$ for $k < 0$.

A two-step approach is used to solve for the coefficients a_k and b_k . In the first step, solving for the coefficients a_k , we use the last n equations of system (13), which after simple manipulations, may be written in matrix form as:

$$\mathbf{H}_2 \mathbf{a} = -\mathbf{h}_{21} \quad (14)$$

with

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathfrak{R}^n, \quad \mathbf{h}_{21} = \begin{bmatrix} h^\alpha(m+1) \\ h^\alpha(m+2) \\ \vdots \\ h^\alpha(m+n) \end{bmatrix} \in \mathfrak{R}^n$$

$$\mathbf{H}_2 = \begin{bmatrix} h^\alpha(m) & h^\alpha(m-1) & \dots & h^\alpha(m-n+1) \\ h^\alpha(m+1) & h^\alpha(m) & \dots & h^\alpha(m-n+2) \\ \vdots & \vdots & \ddots & \vdots \\ h^\alpha(m+n-1) & h^\alpha(m+n-2) & \dots & h^\alpha(m) \end{bmatrix}$$

where $\mathbf{H}_2 \in \mathfrak{R}^{n \times n}$ is an $n \times n$ nonsymmetric Toeplitz matrix (see [22]). If \mathbf{H}_2 is nonsingular, the coefficients a_k ($k = 1, 2, \dots, n$) are uniquely determined by:

$$\mathbf{a} = -\mathbf{H}_2^{-1} \mathbf{h}_{21} \quad (15)$$

In the second step, the coefficients b_k are found using the first $m+1$ equations of system (13), which may be written in matrix form as:

$$\mathbf{H}_1 \bar{\mathbf{a}} = \mathbf{b} \quad (16)$$

where

$$\bar{\mathbf{a}} = \begin{bmatrix} 1 \\ \mathbf{a} \end{bmatrix} \in \mathfrak{R}^{n+1}, \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathfrak{R}^{m+1}$$

$$\mathbf{H}_1 = \begin{bmatrix} h^\alpha(0) & 0 & \cdots & 0 \\ h^\alpha(1) & h^\alpha(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h^\alpha(m) & h^\alpha(m-1) & \cdots & h^\alpha(m-n) \end{bmatrix} \in \mathfrak{R}^{(m+1) \times (n+1)}$$

In this way, we obtain a perfect match between $h(k)$ and the desired impulse response sequence $h^\alpha(k)$ for the first $m+n+1$ values of k . The success of this method depends strongly on the number of selected model coefficients. Since the design method matches $h^\alpha(k)$ only up to the number of model parameters, the more complex the model, the better is the approximation to $h^\alpha(k)$ for $0 \leq k \leq m+n$. However, in practical applications, this introduces a major limitation of the Padé method because the resulting approximation must contain a large number of poles and zeros (see [23]).

It can be shown that the approximations obtained by the CFE method are identical to those resulting by application of the Padé approximation to power series expansion ($m = n$) (see [24]). Nevertheless, the CFE approach is computationally less expensive than the Padé technique.

4.2 Prony's Method

Prony's method differs from the Padé approximation in the scheme of finding the denominator coefficients a_k ($k = 1, 2, \dots, n$) (see Figure 2). These coefficients are determined by LS minimization of the error $e_p(k) = a_k * h^\alpha(k) - b_k$ (where the symbol * denotes convolution):

$$e_p(k) = h^\alpha(k) + \sum_{l=1}^n a_l h^\alpha(k-l), \quad k = m+1, \dots, N-1 \quad (17)$$

Setting the error $e_p(k) = 0$ in system (17) and writing these equations in matrix form:

$$\mathbf{H}_2 \mathbf{a} = -\mathbf{h}_{21} \quad (18)$$

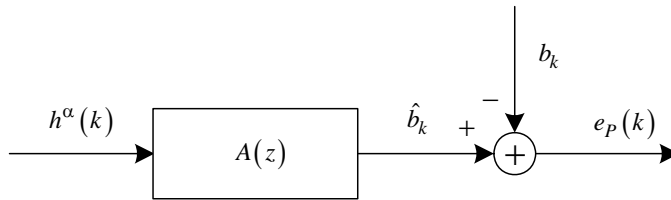


Figure 2
Prony's method

where

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathfrak{R}^n, \quad \mathbf{h}_{21} = \begin{bmatrix} h^\alpha(m+1) \\ h^\alpha(m+2) \\ \vdots \\ h^\alpha(N-1) \end{bmatrix} \in \mathfrak{R}^{N-m-1}$$

$$\mathbf{H}_2 = \begin{bmatrix} h^\alpha(m) & h^\alpha(m-1) & \cdots & h^\alpha(m-n+1) \\ h^\alpha(m+1) & h^\alpha(m) & \cdots & h^\alpha(m-n+2) \\ \vdots & \vdots & \ddots & \vdots \\ h^\alpha(N-2) & h^\alpha(N-3) & \cdots & h^\alpha(N-n-1) \end{bmatrix} \in \mathfrak{R}^{(N-m-1) \times n}$$

It is obvious that in this case system (18) cannot be solved exactly. Therefore, we find the LS solution by solving the set of normal equations:

$$\left(\mathbf{H}_2^T \mathbf{H}_2\right) \mathbf{a} = -\mathbf{H}_2^T \mathbf{h}_{21} \quad (19)$$

If $(\mathbf{H}_2^T \mathbf{H}_2) \in \mathfrak{R}^{n \times n}$ is nonsingular, a unique solution of (19) exists and the coefficients a_k are determined by:

$$\mathbf{a} = -\left(\mathbf{H}_2^T \mathbf{H}_2\right)^{-1} \mathbf{H}_2^T \mathbf{h}_{21} = -\mathbf{H}_2^+ \mathbf{h}_{21} \quad (20)$$

where $\mathbf{H}_2^+ = \left(\mathbf{H}_2^T \mathbf{H}_2\right)^{-1} \mathbf{H}_2^T$ is the pseudoinverse of \mathbf{H}_2 .

With a_k given, the coefficients b_k are found using the Padé method of forcing $h(k) = h^\alpha(k)$ for $k = 0, 1, \dots, m$ (see previous subsection 4.1):

$$b(k) = h^\alpha(k) + \sum_{l=1}^n a_l h^\alpha(k-l), \quad k = 0, 1, \dots, m \quad (21)$$

4.3 Shanks' Method

Shanks' method provides an alternative to Prony's method of finding the numerator coefficients b_k ($k = 0, 1, \dots, m$) (see Figure 3). Thus, instead of forcing an exact fit for the first $m+1$ values of the impulse response sequence, it performs a LS minimization of the error $e_S(k) = h^\alpha(k) - \hat{h}(k)$ over the interval $[0, N-1]$:

$$e_S(k) = h^\alpha(k) - \sum_{l=0}^m b_l g(k-l), \quad k = 0, 1, \dots, N-1 \quad (22)$$

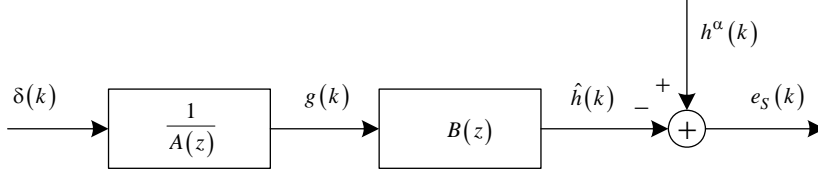


Figure 3
Shanks' method

Firstly, the coefficients a_k are determined in the same way as in Prony's method, that is, by a LS fit over the interval $[m+1, N-1]$ (see previous subsection 4.2). With a_k given, the coefficients b_k are determined following the sequence illustrated in Figure 3:

- 1 Compute the impulse response sequence $g(k)$ of the filter $1/A(z)$ using, for example, the recursion:

$$g(k) = \delta(k) - \sum_{l=1}^n a_l g(k-l), \quad k = 0, 1, \dots, N-1 \quad (23)$$

with $g(k) = 0$ for $k < 0$.

- 2 Solve for the coefficients b_k by setting the error $e_s(k) = 0$ in system (22) and writing these equations in matrix form:

$$\mathbf{G}\mathbf{b} = \mathbf{h} \quad (24)$$

where

$$\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathfrak{R}^{m+1}, \quad \mathbf{h} = \begin{bmatrix} h^\alpha(0) \\ h^\alpha(1) \\ \vdots \\ h^\alpha(N-1) \end{bmatrix} \in \mathfrak{R}^N$$

$$\mathbf{G} = \begin{bmatrix} g(0) & 0 & \cdots & 0 \\ g(1) & g(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g(N-1) & g(N-2) & \cdots & g(N-m-1) \end{bmatrix} \in \mathfrak{R}^{N \times (m+1)}$$

The *LS* solution is found by solving the linear equations:

$$(\mathbf{G}^T \mathbf{G}) \mathbf{b} = \mathbf{G}^T \mathbf{h} \quad (25)$$

If the matrix $(\mathbf{G}^T \mathbf{G}) \in \mathfrak{R}^{(m+1) \times (m+1)}$ is nonsingular, the coefficients a_k can be uniquely determined by:

$$\mathbf{b} = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \mathbf{h} = \mathbf{G}^+ \mathbf{h} \quad (26)$$

where $\mathbf{G}^+ = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T$ is the pseudoinverse of \mathbf{G} .

Note that the Prony and the Shanks methods should yield superior approximations than those obtained by the Padé technique, since $h(k)$ approximates $h^\alpha(k)$, in a least-squares sense, for values of $k > m+n$. Therefore, it will be expected a good match even outside the interval $[0, m+n]$. Also notice that the algorithms just described can be easily evaluated using one of the *LS* solvers available.

5 Illustrative Examples

In this section we use the techniques proposed previously to develop IIR filters approximations of half-differentiators ($s^{1/2}$) and half-integrators ($s^{-1/2}$), sampled at $T = 0.01$ s. For comparison purposes, in the figures that follows, we also plot the IIR filter approximation obtained by the Padé (or the CFE) method for $m = n = 5$.

Figures 4 and 5 depict the Bode diagrams of Prony's approximations to the Tustin and Al-Alaoui operators for $s^{-1/2}$, with $m = n = 5$ and different impulse response lengths of $N = \{11; 100; 200; 500; 1000\}$, respectively. Figures 6 and 7 show the same approximations considering now distinct orders of the IIR filters, namely of $m = n = 1; 3; \dots; 9$ with $N = 1000$. Clearly, the higher the order $m = n$ (or the impulse response length N), of the IIR filter, the better the fitting, in a least-squares sense, to the ideal half-integrator $s^{-1/2}$ (dashed-dotted lines). Note that the Al-Alaoui scheme improves the high frequency magnitude response comparatively to the Tustin scheme. We also verify that the LS approach increases the performance in the low frequency range (corresponding to the steady-state time response) by increasing the order (or the number of impulse values used), resulting in better approximations than those given by the Padé (or the CFE) method. In Figures 8 and 9 are presented the impulse response sequences of the approximations revealing, again, its effectiveness in fitting the discretization schemes of the Tustin and Al-Alaoui operators.

Figures 10 and 11 depict the distribution of poles and zeros of Prony's approximations to the Tustin and Al-Alaoui operators for $s^{-1/2}$, $N = 1000$ and $m = n = \{1; 5; 7; 9\}$. It can be observed that the approximations fulfill the two desired properties: (i) all the poles and zeros lie inside the unit circle, and (ii) the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1, 1)$. Therefore, the resulting approximations are causal, stable and minimum phase, suitable for a digital implementation.

To further illustrate the effectiveness of the proposed techniques, the IIR filters approximations are used to calculate the half-integral and/or the half-derivative of the square and sawtooth functions, as shown in Figures 12 and 13, respectively. Once more, they highlight the effectiveness of the approximations in fitting the ideal curves (dashed-dotted lines).

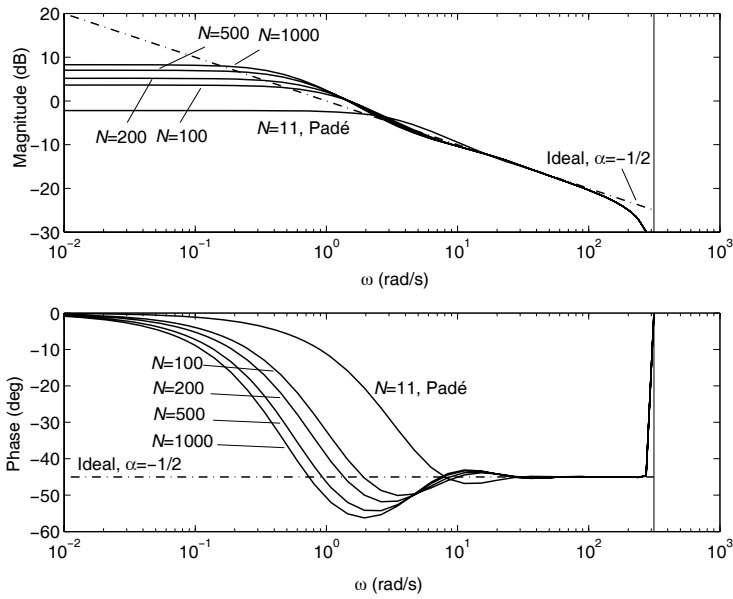


Figure 4

Bode diagrams of Prony's approximations to Tustin operator of $s^{-1/2}$, $m = n = 5$, $T = 0.01$ s and $N = \{11; 100; 200; 500; 1000\}$

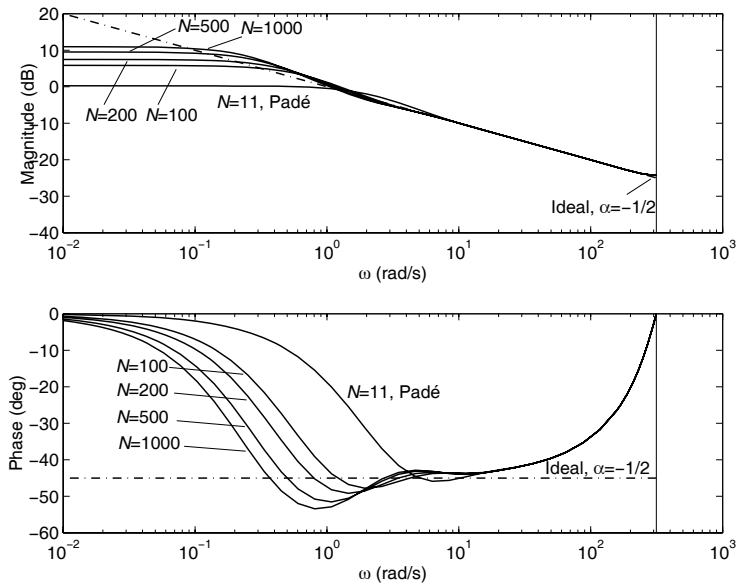


Figure 5

Bode diagrams of Prony's approximations to Al-Alaoui operator of $s^{-1/2}$, $m = n = 5$, $T = 0.01$ s and $N = \{11; 100; 200; 500; 1000\}$

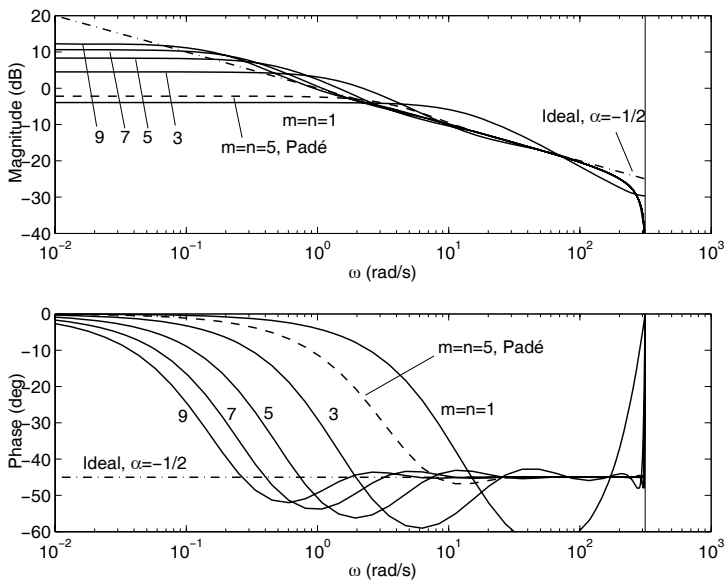


Figure 6

Bode diagrams of Prony's approximations to Tustin operator of $s^{-1/2}$, $N = 1000$, $T = 0.01$ s and $m = n = 1; 3; \dots; 9$

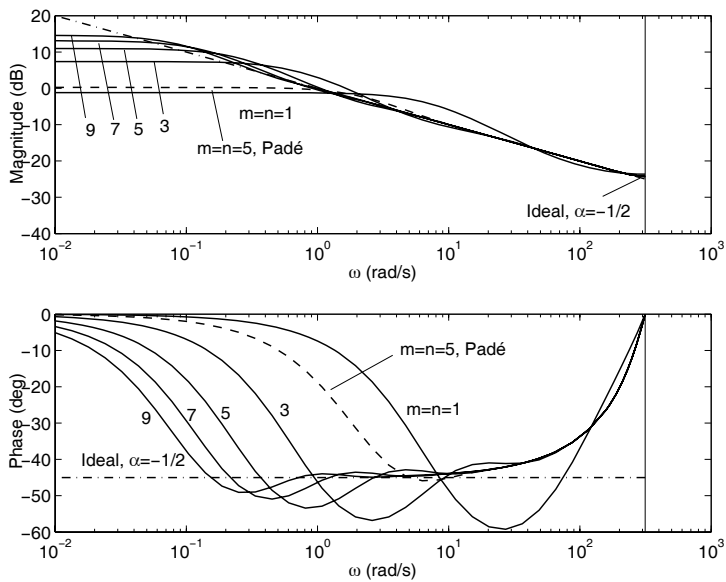


Figure 7

Bode diagrams of Prony's approximations to Al-Alaoui operator of $s^{-1/2}$, $N = 1000$, $T = 0.01$ s and $m = n = 1; 3; \dots; 9$

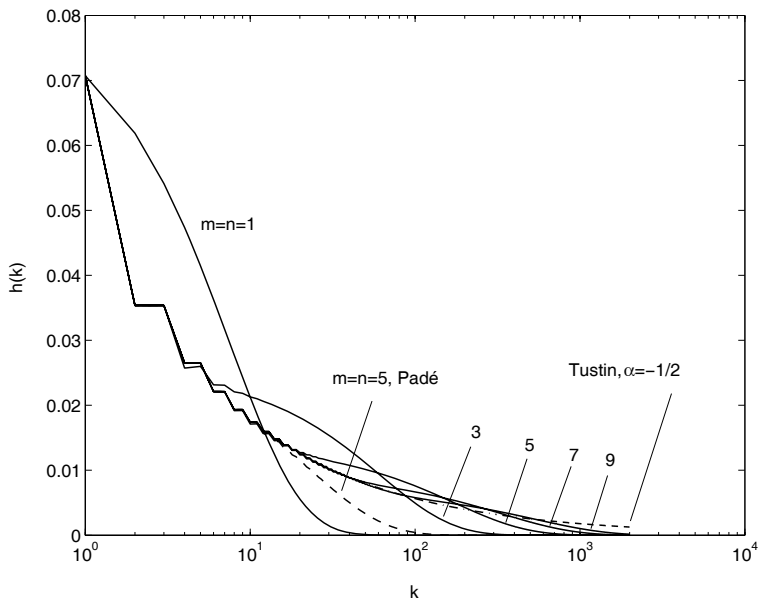


Figure 8

Impulse response sequences of Prony's approximations to Tustin operator of $s^{-1/2}$, $N = 1000$, $T = 0.01$ s and $m = n = 1; 3; \dots; 9$

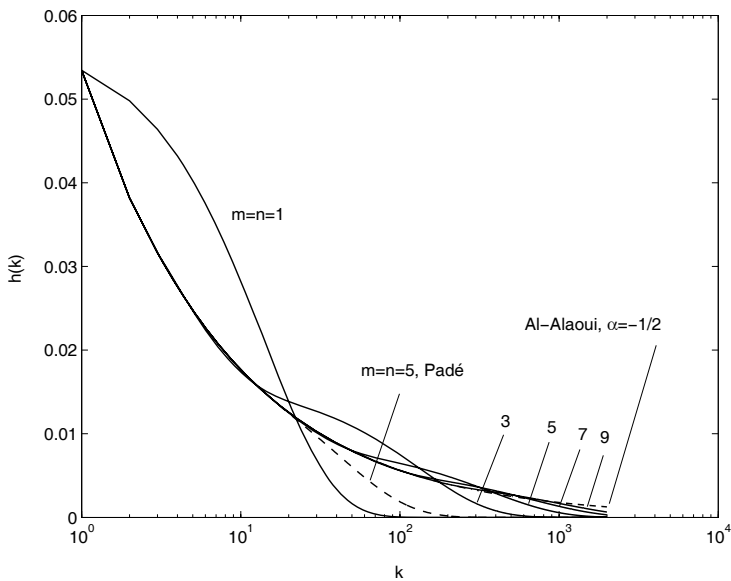


Figure 9

Impulse response sequences of Prony's approximations to Al-Alaoui operator of $s^{-1/2}$, $N = 1000$, $T = 0.01$ s and $m = n = 1; 3; \dots; 9$

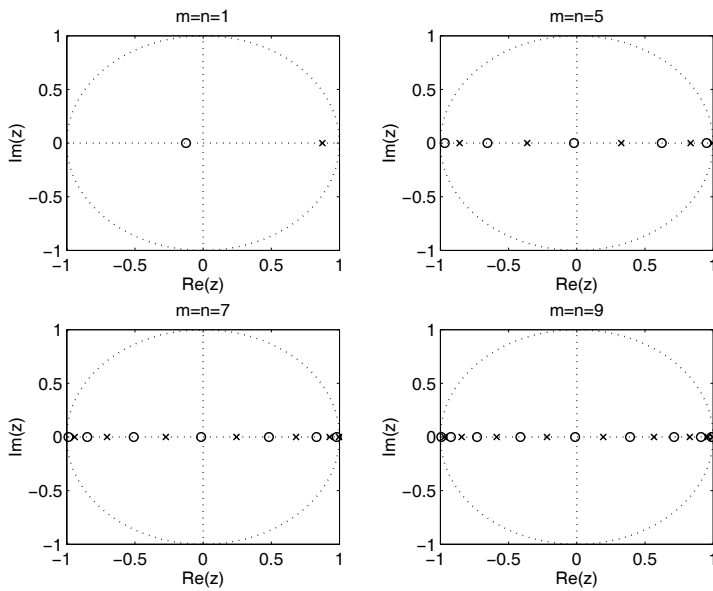


Figure 10

Pole-zero maps of Prony's approximations to Tustin operator of $s^{-1/2}$, $N = 1000$, $T = 0.01$ s and $m = n = \{1; 5; 7; 9\}$

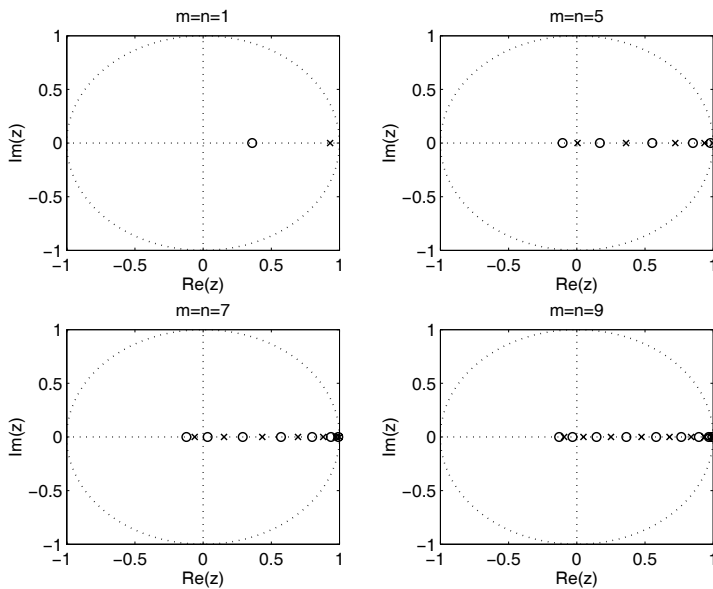


Figure 11

Pole-zero maps of Prony's approximations to Al-Alaoui operator of $s^{-1/2}$, $N = 1000$, $T = 0.01$ s and $m = n = \{1; 5; 7; 9\}$

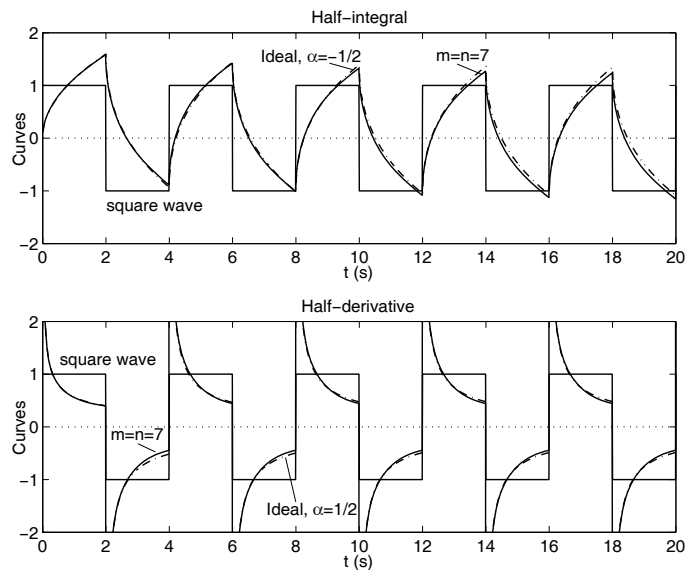


Figure 12

Half-integral/derivative of the square wave function with Shanks' approximation to Euler operator for $N = 1000$, $T = 0.01$ s and $m = n = 7$

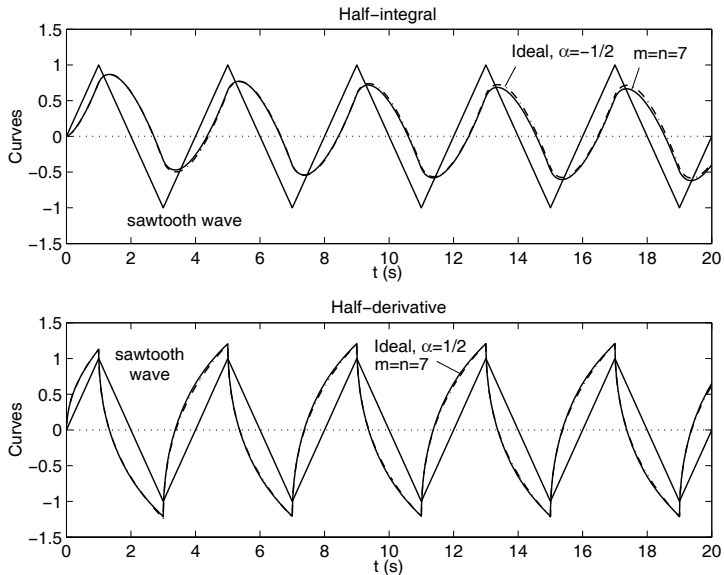


Figure 13

Half-integral/derivative of the sawtooth wave function with Prony's approximation to Euler operator for $N = 1000$, $T = 0.01$ s and $m = n = 7$

Conclusions

In this paper we have described the application of the techniques of Padé, Prony, and Shanks for the design of IIR filters approximations of continuous fractional-order integrators and differentiators of type s^α , $\alpha \in \mathfrak{R}$. The illustrated techniques only require finding the solution of a set of linear equations, yielding good approximations both in the time and the frequency domains. Moreover, it can produce superior approximations than other existent methods, namely the widely used CFE method. Also, the obtained approximations are causal, stable and minimum-phase, suitable for a digital implementation. Some examples are given that demonstrate the effectiveness of the proposed techniques. The results indicate that the least-squares based methods are adequate techniques for obtaining digital approximations of continuous fractional-order operators and, consequently, for the practical realization of fractional-order controllers.

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