Fuzzy Connectives, Residuated Lattices and BL-Algebras

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Abstract: A new method to obtain residuated implications is investigated. In this respect are considered Archimedean conorms and complementation functions having the same generator. It is also defined a basic triple of connectives. Each basic triple induces a residuated lattice and also a BL-algebra.

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1 Introduction

Triangular norms (t-norms) have been introduced in the context of probabilistic metric spaces by Schweizer and Sklar [11].

Definition 1.1. [1] A $t-norm$ is a mapping $T:[0,1]^2 \to [0,1]$, which is associative, commutative, increasing and it satisfies the boundary condition $T(a,1)=a$, for all $a \in [0,1]$.

Definition 1.2. A function C, $C:[0,1] \to [0,1]$ is a complementation function (negation or complement) if and only if C is strictly decreasing, continuous involution satisfying the conditions $C(0)=1, C(1)=0.$

In this paper N denotes the standard negation: $N(a)=1-a$, for all

$a \in [0,1]$.

Definition 1.3. A $t-conorm$ is a function $S:[0,1]^2 \to [0,1]$ such that S is associative, commutative, increasing and it satisfies the boundary condition $S(a,1)=1,$ for all $a \in [0,1]$.

Let C be a complementation function. The $t-norm$ T and the $t-conorm$ S are said to be $C-dual$ if and only if the condition

$$T(C(a),C(b)) = C(S(a,b)), \tag{1}$$

is fulfilled for every $a,b \in [0,1]$.

If T and S are $N-dual$ they are called $dual$.

Example 1.4 The following pairs of $t-norms$ and $t-conorms$ are of particular interest:

i) $T_0(a,b) = \min(a,b),$

$S_0(a,b) = \max(a,b).$

ii) $T_\infty(a,b) = \max(0,a+b-1),$

$S_\infty(a,b) = \min(1,a+b).$

iii) $T_1(a,b) = ab,$

$S_1(a,b) = a+b-ab.$

A $t-norm$ T is said to be $Archimedean$ iff it fulfills the condition $T(a,a) < a,$ for all $a \in (0,1)$.

A $t-conorm$ S is said to be $Archimedean$ iff it fulfills the condition $S(a,a) > a,$ for all $a \in (0,1)$.

Remark 1.5 If T is Archimedean then its C-dual conorm is also Archimedean.

Let f be a continuous and strictly increasing function $f:[u,v] \to [0,\infty]$. The $pseudo-inverse$ of f is the function $f^{(-1)}:[0,\infty] \to [u,v]$, defined by

$$f^{(-1)}(x)=\begin{cases} u, & \textit{if } x\in[0,f(u)] \\ f^{-1}(x), & \textit{if } x\in(f(u),f(v)) \\ v, & \textit{if } x\in[f(v),\infty], \end{cases} \quad (2)$$

where f^{-1} is the ordinary inverse of f.

According to Ling [10], S is an Archimedean $t-conorm$ iff there exists a continuous and strictly increasing function $f:[0,1]\to[0,\infty]$, with $f(0)=0$, such that S may be represented as

$$S(a,b)=f^{(-1)}(f(a)+f(b)), \quad (3)$$

for all $a,b\in[0,1]$.

Moreover, S is strict, i.e., is strictly increasing in $(0,1)$, if and only if $g(1)=+\infty$.

Function f is an $additive\ \ generator$ of S.

A similar result has been stated for complementation function by Trillas [12].

According to [12] C is a complementation function iff C admits the representation

$$C(a)=g^{(-1)}(g(1)-g(a)), \quad (4)$$

for all $a\in[0,1]$ and where the generator $g:[0,1]\to[0,\infty]$, is a continuous and strictly increasing function with $g(0)=0$.

Example 1.6 S_∞ and the standard negation N have the same generator (the identity function on $[0,1]$).

2 Residuated Lattices and BL-Algebras

Definition 2.1 A $residuated\ \ lattice$ ([5], [6]) is an algebraic structure

$L=\langle L,\vee,\wedge,\otimes,\rightarrow,0,1\rangle$, such that:

i) $\langle L,\vee,\wedge,0,1\rangle$ is a lattice with the smallest element 0 and the greatest element 1;

ii) $(L,\otimes,1)$ is an Abelian monoid;

iii) operation $\otimes$ is isotone in both variables;

iv) operation $\rightarrow$ is antitone in the first variable and isotone in the second one;

v) the adjunction condition

$$a\otimes b\leq c \text{ iff } a\leq b\rightarrow c, \tag{5}$$

holds for each $a,\ b,\ c\in L$.

Remark 2.2 i) The operation $\otimes$ is interpreted as the product on L. Sometimes this operation is also called $many-valued\ \ conjunction$, $strong\ \ conjunction$ or $bold\ \ conjunction$ (to differentiate it from the lattice g.l.b. $\wedge$).

ii) The operation $\rightarrow$ is called the $residuum$ (with respect to $\otimes$). From a logical point of view $\rightarrow$ denotes the connective $implication$.

iii) The pair $(\otimes,\rightarrow)$ satisfying the adjunction property v) is said to be an $adjoin\ \ couple$.

Definition 2.3 Consider the structure $\langle L,\vee,\wedge,\otimes,0,1\rangle$, where $\otimes$ is the product in L. Define $I(a,b)=\vee\{x\mid a\otimes x\leq b\}$.

$I(a,b)$ is the $residuated\ \ implication$ generated by the product $\otimes$ [3]. The pair $(\otimes,I)$ is called an $adjoin\ \ couple$.

Remark 2.4 Usually a residuated lattice is enriched by an unary antitone and involutive operation. This operation may be interpreted as the connective negation.

Definition 2.5 A $distributive\ \ residuated\ \ lattice$ is an algebraic structure $L=\langle L,\vee,\wedge,\otimes,\rightarrow,0,1\rangle$, such that:

i) $\langle L,\vee,\wedge,\otimes,\rightarrow,0,1\rangle$ is a residuated lattice;

ii) $\langle L,\vee,\wedge,0,1\rangle$ is a distributive lattice.

Example 2.6 Assume $L=[0,1]$ and put

$$a\otimes b=T_{\infty}(a,b),$$

$$a\wedge b=T_0(a,b),$$

$$a\vee b=S_0(a,b),$$

$$a\to b=T_0(1,1-a+b),$$

where $T_{\infty}(a,b)$ is the Lukasiewicz product and $T_0(1,1-a+b)$ is the Lukasiewicz implication.

The Lukasiewicz-type structure $L=\langle[0,1],S_0,T_0,T_{\infty},\to,0,1\rangle$, is a residuated lattice.

It is interesting to note that the Lukasiewicz implication may also be written as $a\to b=S_{\infty}(N(a),b)$.

Indeed we have $S_{\infty}(N(a),b)=\min(1,1-a+b)$.

Example 2.7 Let T be a continuous $t-norm$. Denote by I_T the residuated implication generated by T (also called $T-implication$):

$$I_T(a,b)=\vee\{c\in[0,1]\,|\,T(a,c)\le b\}.$$

The system $\langle[0,1],S_0,T_0,T,I_T,0,1\rangle$, is a residuated lattice.

For $T=T_0$ we obtain the implication $I_{T_0}(a,b)=\begin{cases}1, & if\ a\le b,\\ b, & otherwise.\end{cases}$

The residuated implication generated by $T=T_1$ is

$$I_{T_1}(a,b)=\begin{cases}1, & if\ a\le b,\\ \dfrac{b}{a}, & otherwise.\end{cases}$$

It is interesting to note that residuated implication generated by T_{∞} is just the

Lukasiewicz implication.

Enriching a residuated lattice with some axioms we obtain a BL-algebra as follows:

Definition 2.8 A $BL-algebra$ [4] is an algebraic structure $L=\langle L,\vee,\wedge,\otimes,\rightarrow,0,1\rangle$, such that:

i) $\langle L,\vee,\wedge,\otimes,\rightarrow,0,1\rangle$ is a distributive residuated lattice;

ii) $x\wedge y=x\otimes(x\rightarrow y)$;

iii) $(x\rightarrow y)\vee(y\rightarrow x)=1$.

Remark 2.9 It is easy to see that the residuated lattice considered in Example 2.6 is also a BL-algebra.

3 Basic Triples

Let $L(X)$ be the family of the fuzzy sets on the universe X. Consider the intersection, union and complement of fuzzy sets are induced by a certain triple (T,S,C). We assume that T and S are C-dual. We may now ask about a compatibility condition for C.

We would suppose that the coherence of the obtained fuzzy theory depends on the internal compatibility of the generating triple (T,S,C). One way to study the compatibility of C with S and T is to use the generators. We recall that a conorm and a complement may have the same generator. It seems thus natural to assume that (T,S,C) has the maximum compatibility if S and C has the same generator and T, S are C-dual.

We are lead to the following definitions.

Definition 3.1 The conorm S and the complementation function C are called $f-generated$ if and only if they have the same generator f.

Definition 3.2 The system (T,S,C) is said to be a basic triple [8] if and only if the following requirements are fulfilled:

i) T is an Archimedean t-norm;

ii) C is a complementation function;

iii) S is the conorm C-dual of T;

iv) S and C are f -generated.

Example 3.3 $\left(T_{\infty}, S_{\infty}, N\right)$ is a basic triple.

Moreover this is the very unique triple of the form $\left(T, S, N\right)$. Therefore this basic triple seems to have a special position in fuzzy set theory. We may recall, for instance, that the fuzzy operations induced by T_{∞} and S_{∞} are the unique ones such that a fuzzy partition of the universe is also a partition of unity [7].

4 Matching Operator

The matching operator has been introduced in [8] (see also [9]).

Definition 4.1 Let S be an Archimedean conorm and C a complementation function. The *matching operator* generated by S and C, $m:[0,1]^2 \to [0,1]$ is defined by $m(a,b) = S\left(C(a), b\right)$, for all a, $b \in [0,1]$.

Remark 4.2 Since S is commutative we have $m(b,a) = S\left(C\left(C(a)\right), C(b)\right)$, and thus $m(b,a) = m\left(C(a), C(b)\right)$, for all a, $b \in [0,1]$.

Using the properties of the t-conorm S we obtain

Proposition 4.3 Let S be an Archimedean conorm and C is a complementation function. Consider S and C have the same generator. The matching operator generated by S and C fulfills the following properties:

i) $m(a,b) = f^{(-1)}\left(f(1) - f(a) + f(b)\right)$;

ii) $m(a,b) \geq C(a)$;

iii) $m(0,a) = 1$,

iv) $m(a,0) = C(a)$,

v) $m(1,a) = a$,

vi) $m(a,1)=1,$

vii) $m(a,a)=1,$

viii) if $a<b,$ then $m(a,b)=1,$

for all $a,b\in[0,1].$

Proof. i) From Definition 4.1 we have $m(a,b)=S(C(a),b).$

Using equalities (3) and (4) we obtain that

$$m(a,b)=f^{(-1)}\left(f\left(C(a)\right)+f(b)\right)$$
$$=f^{(-1)}\left(f\left(f^{(-1)}\left(f(1)-f(a)\right)\right)+f(b)\right).$$

Thus we have

$$m(a,b)=f^{(-1)}\left(f(1)-f(a)+f(b)\right). \tag{6}$$

ii) Using Definition 4.1 and since S is a increasing function we obtain that

$$m(a,b)\geq S\left(C(a),0\right). \tag{7}$$

Using equality (3) we have that

$$S\left(C(a),0\right)=f^{(-1)}\left(f\left(C(a)\right)+f(0)\right)$$

$$=f^{(-1)}\left(f\left(f^{(-1)}\left(f(1)-f(a)\right)\right)+0\right)$$

$$=f^{(-1)}\left(f\left(C(a)\right)\right)$$

$$=C(a). \tag{8}$$

From inequality (7) and equality (8) we obtain that $m(a,b)\geq C(a).$

By using equality (6) and definition (2) of pseudo-inverse of f we may write.

iii) $m(0,a)=f^{(-1)}\left(f(1)-f(0)+f(a)\right) \;=f^{(-1)}\left(f(1)+f(a)\right).$

Since $f(1)+f(a)\geq f(1),$ it follows that

$f^{(-1)}\left(f(1)+f(a)\right)=f^{(-1)}\left(f(1)\right).$

Thus we have $m(0,a)=1$.

iv) Using equation (6) we have $m(a,0)=f^{(-1)}\left(f(1)-f(a)+f(0)\right)$

$=f^{(-1)}\left(f(1)-f(a)\right).$

Using equation (4) we obtain $m(a,0)=C(a)$.

v) From equation (6) we have that $m(1,a)=f^{(-1)}\left(f(1)-f(1)+f(a)\right)$

$=f^{(-1)}\left(f(a)\right)\ =a.$

vi) By putting $b=1$ in equation (6) we obtain

$m(a,1)=f^{(-1)}\left(f(1)-f(a)+f(1)\right).$

Since $f(a)\leq f(1)$, it follows that $f(1)-f(a)+f(1)\geq f(1)$.

Therefore we have $m(a,1)=f^{(-1)}(1)\ =1$.

vii) If we put $b=a$ in equation (6) we have

$m(a,a)=f^{(-1)}\left(f(1)-f(a)+f(a)\right)\ =f^{(-1)}\left(f(1)\right)\ =1.$

viii) Consider again equation (6): $m(a,b)=f^{(-1)}\left(f(1)-f(a)+f(b)\right)$.

If $a<b$ we have that $f(a)<f(b)$.

Therefore $f(1)-f(a)+f(b)\geq f(1)$, and thus we have $m(a,b)=f^{(-1)}(1)\ =1$.

This completes the proof. □

Remark 4.4 It is easy to see that we have re-obtained some well known properties of the Lukasiewicz implication.

5 Matching Operator and Residuated Implication

According to [8] (see also [9]) a residuated implication and an adjoin couple may be generated by a matching operator. In this respect we have the following theorem.

Theorem 5.1 Let S be an Archimedean comorm and C is a complementation function. Consider S and C have the same generator. The matching operator m generated by S and C is a residuated implication.

Proof. From Definition 4.1 and equality (1) we have that

$$T(a,x) = C\Big(m\big(a,C(x)\big)\Big). \tag{9}$$

Thus the residuated implication

$$I(a,b) = \vee\{x \mid a \otimes x \leq b\} = \vee\Big\{x \mid C\Big(m\big(a,C(x)\big)\Big) \leq b\Big\}.$$

In what follows we will solve the inequality

$$C\Big(m\big(a,C(x)\big)\Big) \leq b. \tag{10}$$

Since the complementation function C is decreasing, the inequality (10) is equivalent with

$$m\big(a,C(x)\big) \geq C(b). \tag{11}$$

Using Proposition 4.3 i) and equality (4) we obtain that

$$m\big(a,C(x)\big) = f^{(-1)}\Big(f(1) - f(a) + f\big(C(x)\big)\Big)$$

$$= f^{(-1)}\big(f(1) - f(a)\big) + f\Big(f^{(-1)}\big(f(1) - f(x)\big)\Big).$$

Thus we have

$$m\big(a,C(x)\big) = f^{(-1)}\big(f(1) - f(a) + f(1) - f(x)\big). \tag{12}$$

Using equalities (4) and (12) the inequality (11) becomes

$$f^{(-1)}\big(f(1) - f(a) + f(1) - f(x)\big) \geq f^{(-1)}\big(f(1) - f(b)\big). \tag{13}$$

To solve this inequality we have two different situations:

i) Consider the case $x \leq C(a)$.

Using equality (4) we obtain that $f(x) \le f^{(-1)}\left(f(1) - f(a)\right)$, which is equivalent to $f(1) \ge f(a) + f(x)$.

Using Definition 1.2 of the pseudoinverse function $f^{(-1)}$, the inequality (13) becomes $f(1) \ge f(1) - f(b)$, which is equivalent to $f(b) \ge 0$.

This inequality holds for all

$$x \in \left[0, C(a)\right]. \tag{14}$$

ii) Let us now consider the case $x > C(a)$.

Inequality (13) becomes $f(1) - f(a) + f(1) - f(x) \ge f(1) - f(b)$, which is equivalent to $f(x) \le f(1) - f(a) + f(b)$.

It follows that $x \le f^{(-1)}\left(f(1) - f(a) + f(b)\right)$.

Using Proposition 4.3 i) we obtain that $x \le m(a,b)$.

From Proposition 4.3 ii) we obtain that

$$x \in \left(C(a), m(a,b)\right]. \tag{15}$$

From relations (14) and (15) it follows that the solution of the inequality (10) is $x \in \left[0, C(a)\right] \cup \left(C(a), m(a,b)\right]$, and thus the Definition 2.3 can be written as $I(a,b) = \vee\left\{x \mid x \in \left[0, m(a,b)\right]\right\}$.

Therefore we have that $I(a,b) = m(a,b)$.

This completes the proof. □

6 Residuated Lattices Generated by a Regular Basic Triple

Let (T,S,C) be a regular basic triple. A method for generating a residuated lattice from (T,S,C) has been proposed in [8] (see also [9]).

Like in MV-algebras [2] (see [8], [9]) we may define the operations $\wedge$ and $\vee$

induced by T and S as: $a \wedge b = T\left(S\left(a, C\left(b\right)\right), b\right)$, $a \vee b = S\left(T\left(a, C\left(b\right)\right), b\right)$, for all $a,\ b \in \left[0,1\right]$.

Since T and S are C-dual we may write

$T\left(a, C\left(b\right)\right) = T\left(C\left(C\left(a\right)\right), C\left(b\right)\right) = C\left(S\left(C\left(a\right), b\right)\right) = C\left(m\left(a,b\right)\right)$.

Thus we have $a \vee b = S\left(T\left(a, C\left(b\right)\right), b\right) = S\left(C\left(m\left(a,b\right)\right), b\right)$.

Therefore we obtain

$$a \vee b = m\left(m\left(a,b\right), b\right), \tag{16}$$

for all $a,\ b \in \left[0,1\right]$.

This equality generalizes the definition of $\vee$ in the Lukasiewicz logic:

$p \vee q = \left(p \to q\right) \to q.$

It is easy to see that

$$a \wedge b = T\left(m\left(b,a\right), b\right), \tag{17}$$

for all $a,\ b \in \left[0,1\right]$.

We may also write $a \wedge b$ as follows: $a \wedge b = T\left(S\left(a, C\left(b\right)\right), b\right)$

$= C\left(S\left(C\left(S\left(a, C\left(b\right)\right)\right), C\left(b\right)\right)\right) = C\left(S\left(C\left(m\left(b,a\right)\right), C\left(b\right)\right)\right)$

$= C\left(m\left(m\left(b,a\right), C\left(b\right)\right)\right)$.

Since $m\left(b,a\right) = m\left(C\left(a\right), C\left(b\right)\right)$, we obtain

$$a \wedge b = C\left(m\left(m\left(C\left(a\right), C\left(b\right)\right), C\left(b\right)\right)\right). \tag{18}$$

Therefore we re-obtained definition of $\wedge$ in the Lukasiewicz logic, i.e.

$p \wedge q = N\left(\left(N\left(p\right) \to N\left(q\right)\right) \to N\left(q\right)\right),$

where $N\left(p\right)$ is a negation of proposition p.

Remark 6.1 It is easy to see that $C\left(a \wedge b\right) = C\left(a\right) \vee C\left(b\right)$, i.e., De Morgan laws also hold.

A surprising result concerning lattice operations induced by a regular basic triple is given by the next:

Proposition 6.2 Lattice operations induced by any regular basic triple (T,S,C) via definitions (16) and (18) are reduced to $\max$ and $\min$:

i) $a \vee b = \max(a,b)$,

ii) $a \wedge b = \min(a,b)$.

Proof. i) Let consider equality (16) $a \vee b = m(m(a,b),b)$.

We have three different situations:

a) if $a < b$, from Proposition 4.3 viii) we have that $a \vee b = m(1,b)$.

Using Proposition 4.3 v) we obtain

$$a \vee b = b. \tag{19}$$

b) if $a = b$, from Proposition 4.3 vii) we obtain $a \vee b = m(b,b)$,

and thus we have

$$a \vee b = b. \tag{20}$$

c) if $a > b$, from Proposition 4.3 i) we have that

$$a \vee b = f^{(-1)}\left(f(1) - f\left(f^{(-1)}\left(f(1) - f(a) + f(b)\right)\right) + f(b)\right).$$

Since $a > b$ is equivalent with $f(a) > f(b)$,

using definition (2) of pseudo-inverse of f we obtain that

$$a \vee b = f^{(-1)}\left(f(1) - f(1) + f(a) - f(b) + f(b)\right) = f^{(-1)}\left(f(a)\right),$$

and thus we have

$$a \vee b = a. \tag{21}$$

From equalities (19), (20) and (21) it follows that $a \vee b = \max(a,b)$.

ii) Using equality (18) we have that $a \wedge b = C\left(C(a) \vee C(b)\right)$.

We have three different situations:

a) if $a < b$ we have that $C(a) > C(b)$, and then we obtain $a \wedge b = C(C(a))$.

Thus we have

$$a \wedge b = a. \tag{22}$$

b) if $a = b$ we have that $C(a) = C(b)$, and then we obtain

$$a \wedge b = a. \tag{23}$$

c) if $a > b$ we have that $C(a) < C(b)$, and then we obtain $a \wedge b = C(C(b))$, and thus we have

$$a \wedge b = b. \tag{24}$$

From equalities (22), (23) and (24) it follows that $a \wedge b = \min(a,b)$.

This completes the proof. □

Remark 6.3 From Proposition 6.2 easy follows that $L = \langle [0,1], \wedge, \vee, 0, 1 \rangle$, is a bounded distibutive lattice.

Now we are able to state the next result.

Theorem 6.4 Let (T,S,C) be a regular basic triple and m is the matching operator generated by S and C. Then the system $L = \langle [0,1], \wedge, \vee, T, m \rangle$ is a residuated lattice. Moreover (T,m) is an adjoin couple of the residuated lattice L.

Proof. i) Use Remark 6.3.

ii) Using definition of t-norm T is obvious that T is isotone in both variables and $(L, \otimes, 1)$ is an Abelian monoid.

iii) Using Definition 2.1, definition of t-conorm S and definition of complementation function is easy to see that m is antitone in first variable and isotone in the second one.

Therefore the first four axioms of Definition 2.1 are fulfilled.

iv) We have to prove that the adjuction condition (5)

$T(a,b) \le c \Leftrightarrow a \le m(b,c)$, also holds.

Let us assume that the inequality $T(a,b) \le c,$ holds.

Using equalities (9), (4) and Proposition 4.3 i) we have that

$f^{(-1)}\left(f(a)+f(b)-f(1)\right) \le c,$ which is equivalent to

$$f\left(f^{(-1)}\left(f(a)+f(b)-f(1)\right)\right) \le f(c). \quad (25)$$

We have three different situations:

a) if $0 < f(a)+f(b)-f(1) < 1$, it follows that inequality (25) is equivalent to $f(a)+f(b)-f(1) \le f(c)$.

From this inequality we obtain successively $f(a) \le f(1)-f(b)+f(c),$

$f^{(-1)}\left(f(a)\right) \le f^{(-1)}\left(f(1)-f(b)+f(c)\right).$

Using equality (6) we have that $a \le m(b,c)$.

b) if $f(a)+f(b)-f(1) \ge 1$ inequality (25) becomes $f(1) \le f(c)$.

Since $c \in [0,1]$ and f is increasing it follows that $c = 1$.

From Proposition 4.3 vi) we have $a \le 1 \; = m(b,c)$.

c) if $f(a)+f(b)-f(1) \le 0,$ since f is a increasing function it follows that

$f(a)+f(b)-f(1) \le f(c)$.

From $f(a) \le f(1)-f(b)+f(c),$ we have $f^{(-1)}\left(f(a)\right) \le f^{(-1)}\left(f(1)-f(b)+f(c)\right).$

Using Proposition 4.3 i) we obtain that $a \le m(b,c)$.

Thus we proved that in all situations the equivalence

$T(a,b) \le c \Leftrightarrow a \le m(b,c),$ expressing the adjunction condition, holds.

It follows that (T,m) is an adjoin couple and the system $L = \langle [0,1], \wedge, \vee, T, m \rangle,$ is a residuated lattice.

This completes the proof. □

Remark 6.5 i) From Theorem 6.4 and Remark 6.3 easily follows that the system $L = \langle [0,1], \wedge, \vee, T, m \rangle$, is a distributive residuated lattice.

ii) Residuated lattice L may be enriched with a complement operation C.

7 BL-Algebras Generated by a Regular Basic Triple

In the previous section we have proved that using any regular basic triple (T,S,C) we can obtain an algebraic system which is a distributive residuated lattice. A BL-algebra is a residuated lattice having some extra axioms. Naturally occur the question, if using any regular basic triple (T,S,C), can we obtain an algebraic system which to be a BL-algebra.

This section is presenting the answer to the above question. It is given in the main result of this paper, stated in the next:

Theorem 7.1 Let (T,S,C) be a basic triple and m the matching operator generated by S and C. Then $L = \langle [0,1], \wedge, \vee, T, m \rangle$ is a BL-algebra.

Proof. We prove that the axioms of Definition 2.8 are fulfilled.

i) From Remark 6.5 we have that L is a distributive residuated lattice.

ii) Any t-norm T is commutative. Thus we have that

$$T(a, m(a,b)) = T(m(a,b), a).$$

From equality (17) we have that $T(a, m(b,a)) = b \wedge a$

Since operation $\wedge$ is commutative, it follows that $T(a, m(b,a)) = a \wedge b,$ i.e., the second axiom of Definition 2.8 holds.

iii) It is well known that if $x \leq y$ we have that

$$x \vee y = y. \tag{26}$$

If $a \leq b$, from Proposition 4.3 viii) it follows that

$$m(a,b) \vee m(b,a) = 1 \vee m(b,a).$$

Using equality (26) we obtain

$$m(a,b) \vee m(b,a) = 1. \quad (27)$$

If $a > b$, form Proposition 4.3 viii) it follows that $m(a,b) \vee m(b,a) = m(b,a) \vee 1$.

Using equality (26) we obtain

$$m(a,b) \vee m(b,a) = 1. \quad (28)$$

From equalities (27) and (28) it follows that $m(a,b) \vee m(b,a) = 1$, for all $a,b \in [0,1]$.

This completes the proof. □

Conclusions

This paper can be considered an extension of [9]. So, here are proved all the results which were only presented in the above mentioned paper. Also, new concepts are introduced and new properties are proved.

The role of basic triples in defining set operations for fuzzy sets is investigated. Basic triples are used for introducing a matching operator. It is proved that the matching operator represents a residuated implication.

We have also proved that a basic triple induces a residuated lattice and moreover that induces a BL-algebra.

The method of obtaining a BL-algebra from a basic triple is the main result of this paper.

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