Componentwise Stability of BAM Neural Networks with Uncertainties

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Abstract: The componentwise (exponential) asymptotic stability, abbreviated as CW(E)AS, is a special type of asymptotic stability which ensures the individual monitoring of each state-space variable of a dynamical system. Our paper provides sufficient conditions for the CW(E)AS analysis of Bidirectional Associative Memory (BAM) neural networks with uncertainties both in the parameters and in the activation functions. These conditions are formulated in terms of Hurwitz stability of a test matrix built from the information available about the uncertainties affecting the dynamics of the considered BAMs. Some interesting results are derived as particular cases, which allow comparisons with several other works addressing the stability of Hopfield neural networks.

Keywords: bidirectional associative memory, componentwise (exponential) asymptotic stability, robustness

1 Introduction

Consider the *bidirectional associative memory* (BAM) neural network without delay described by

$$\begin{aligned} \dot{x}^1(t) &= A^1 x^1(t) + W^2 f^2\left(x^2(t)\right) + I^1, \quad x^1(t_0) = x_0^1 \in \Re^m \\ \dot{x}^2(t) &= A^2 x^2(t) + W^1 f^1\left(x^1(t)\right) + I^2, \quad x^2(t_0) = x_0^2 \in \Re^n \end{aligned} \tag{1}$$

where $\boldsymbol{A}^1 = \mathrm{diag}\{a_1^1, a_2^1, \ldots, a_m^1\}$, $\boldsymbol{A}^2 = \mathrm{diag}\{a_1^2, a_1^2, \ldots, a_n^2\}$, $\boldsymbol{W}^1 = \left[w_{ji}^1\right]$, $\boldsymbol{W}^2 = \left[w_{ij}^2\right]$ are matrices of appropriate sizes, $\boldsymbol{x}^1 = \left[x_1^1, x_2^1, \ldots, x_m^1\right]^\tau$, $\boldsymbol{x}^2 = \left[x_1^2, x_2^2, \ldots, x_n^2\right]^\tau$ are the state vectors and $\boldsymbol{I}^1 = \left[I_1^1, I_2^1, \ldots, I_m^1\right]^\tau$, $\boldsymbol{I}^2 = \left[I_1^2, I_2^2, \ldots, I_n^2\right]^\tau$ are the input vectors (τ denoting the vector transposition).

All the components of the activation functions $f^1 : \Re^m \to \Re^m$, $\boldsymbol{f}^1(\boldsymbol{x}^1) = \left[f_1^1(x_1^1), f_2^1(x_2^1), \ldots, f_m^1(x_m^1)\right]^\tau$, and $f^2 : \Re^n \to \Re^n$, $\boldsymbol{f}^2(\boldsymbol{x}^2) = \left[f_1^2(x_1^2), f_2^2(x_2^2), \ldots, f_n^2(x_n^2)\right]^\tau$ are nondecreasing and globally Lipschitz continuous, i.e. for all $i = \overline{1,m}$, $j = \overline{1,n}$, there exist $L_i^1, L_j^2 > 0$ so that

$$0 \le \frac{f_i^1(r) - f_i^1(s)}{r - s} \le L_i^1, \quad 0 \le \frac{f_j^2(r) - f_j^2(s)}{r - s} \le L_j^2, \tag{2}$$

for all $r, s \in \Re, r \neq s$. Obviously, these hypotheses on $\boldsymbol{f}^1$ and $\boldsymbol{f}^2$ ensure that the Cauchy problem (1) has a unique solution $\boldsymbol{x}^1(t) = \boldsymbol{x}^1(t; t_0, \boldsymbol{x}_0^1, \boldsymbol{x}_0^2)$, $\boldsymbol{x}^2(t) = \boldsymbol{x}^2(t; t_0, \boldsymbol{x}_0^1, \boldsymbol{x}_0^2)$, which is defined for all $t \ge t_0 \ge 0$. Moreover, (2) implies that if f is continuously differentiable on $\Re$, then its derivative satisfies $0 \le df(s)/ds \le L$ for all $s \in \Re$.

Recent papers, such as [1] – [4], provide sufficient conditions, formulated in algebraic terms, for the global (exponential) asymptotic stability of BAMs. These results are presented in the broader context of delayed states, but they are also applicable to BAM (1).

For many problems encountered in practice it is important to consider that the entries of the matrices $\boldsymbol{A}^k$, $\boldsymbol{W}^k$, $k = \overline{1,2}$, defining the dynamics of BAM (1), are uncertain, in the sense of the matrix componentwise inequalities:

$$\begin{aligned}
&\underline{\boldsymbol{A}}^1 = \operatorname{diag}\left\{\underline{a}_1^1, \underline{a}_2^1, \ldots, \underline{a}_m^1\right\} \le \boldsymbol{A}^1 \le \overline{\boldsymbol{A}}^1 = \operatorname{diag}\left\{\overline{a}_1^1, \overline{a}_2^1, \ldots, \overline{a}_m^1\right\}, \\
&\underline{\boldsymbol{A}}^2 = \operatorname{diag}\left\{\underline{a}_1^2, \underline{a}_2^2, \ldots, \underline{a}_n^2\right\} \le \boldsymbol{A}^2 \le \overline{\boldsymbol{A}}^2 = \operatorname{diag}\left\{\overline{a}_1^2, \overline{a}_2^2, \ldots, \overline{a}_n^2\right\}, \\
&\underline{\boldsymbol{W}}^1 = \left[\underline{w}_{ji}^1\right]_{\substack{j=\overline{1,n} \\ i=\overline{1,m}}} \le \boldsymbol{W}^1 \le \overline{\boldsymbol{W}}^1 = \left[\overline{w}_{ji}^1\right]_{\substack{j=\overline{1,n} \\ i=\overline{1,m}}}, \\
&\underline{\boldsymbol{W}}^2 = \left[\underline{w}_{ij}^2\right]_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}} \le \boldsymbol{W}^2 \le \overline{\boldsymbol{W}}^2 = \left[\overline{w}_{ij}^2\right]_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}.
\end{aligned} \tag{3}$$

Consequently, let us introduce the following classes of matrices:

$$\mathrm{A}^1 = \left\{ A^1 \in \Re^{mxm} \,\middle|\, \underline{A}^1 \le A^1 \le \overline{A}^1 \right\}$$

$$\mathrm{A}^2 = \left\{ A^2 \in \Re^{nxn} \,\middle|\, \underline{A}^2 \le A^2 \le \overline{A}^2 \right\} \tag{4}$$

$$\mathcal{W}^1 = \left\{ W^1 \in \Re^{nxm} \middle| \underline{W}^1 \le W^1 \le \overline{W}^1 \right\},$$

$$\mathcal{W}^2 = \left\{ W^2 \in \Re^{mxn} \middle| \underline{W}^2 \le W^2 \le \overline{W}^2 \right\}$$

When discussing the uncertainties that can affect the dynamics of BAM (1), it is also natural to take into consideration the classes of activation functions $\mathcal{F}^1$ and $\mathcal{F}^2$ defined by positive vectors $\lambda^1 = \left[\lambda_1^1, \lambda_2^1, \ldots, \lambda_m^1\right]^T > \mathbf{0}$ and $\lambda^2 = \left[\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2\right]^T > \mathbf{0}$ as follows:

$$\inf\left\{ L_i^1 > 0 \mid 0 \le \frac{f_i^1(r) - f_i^1(s)}{r - s} \le L_i^1, \forall r, s \in \Re, r \ne s \right\} \le \lambda_i^1$$

$$\inf\left\{ L_j^2 > 0 \mid 0 \le \frac{f_j^2(r) - f_j^2(s)}{r - s} \le L_j^2, \forall r, s \in \Re, r \ne s \right\} \le \lambda_j^2 \qquad (5)$$

for $i = \overline{1,m}$, $j = \overline{1,n}$, ensuring desired bounds for the slope of each component of the vector functions $\boldsymbol{f}^k$, $k = \overline{1,2}$.

This paper proves that the stability of a single test matrix guarantees a stronger stability property of BAM (1), called *componentwise stability*, for all $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k = \overline{1,2}$. Unlike the standard concepts of stability, that give global information on the state-space vector, expressed in terms of arbitrary norms, the componentwise stability allows an individual monitoring of each state-space variable. This type of stability was first studied by Voicu in [5] who applied the theory of flow-invariant time-dependent rectangular sets to define and characterize the *componentwise asymptotic stability* (CWAS) and the *componentwise exponential asymptotic stability* (CWEAS) for continuous-time linear systems. Further works extended the analysis of componentwise stability to continuous-time delay linear systems [6], 1-D and 2-D discrete-time linear systems [7], interval matrix systems [8] and a class of Persidskii systems with uncertainties [9]. Despite the existence of these results, the componentwise stability of recurrent neural networks remained almost unexplored, except for a reduced number of recent papers [10]–[13].

Our paper develops a robustness analysis for CWAS/CWEAS of BAM (1) with respect to both the parameters (in the sense that $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $k = \overline{1,2}$) and the activation functions $\boldsymbol{f}^k \in \mathcal{F}^k$, $k = \overline{1,2}$. The concepts employed by our work are rigorously defined in Section II. Section III provides the main results, consisting in sufficient criteria for the CWAS/CWEAS of BAMs with uncertainties. Section IV creates a deeper insight into some frequently encountered particular cases and allows comparisons with other papers. A few final remarks are formulated in Section V. All over the text, the vector (matrix) inequalities have componentwise meaning.

2 Preliminaries

Assume that BAM (1) has a finite number of equilibrium points and let $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ be one of these, i.e. $\boldsymbol{A}^1\boldsymbol{x}_e^1 + \boldsymbol{W}^2\boldsymbol{f}^2\left(\boldsymbol{x}_e^2\right) + \boldsymbol{I}^1 = \boldsymbol{0}$ and $\boldsymbol{A}^2\boldsymbol{x}_e^2 + \boldsymbol{W}^1\boldsymbol{f}^1\left(\boldsymbol{x}_e^1\right) + \boldsymbol{I}^2 = \boldsymbol{0}$.

Definition 1. (**a**) Let $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ be two vector functions $p^1 : \Re_+ \to \Re^m$, $p^2 : \Re_+ \to \Re^n$, continuously differentiable, with positive components $p_i^1(t) > 0$, $i = \overline{1,m}$, $p_j^2(t) > 0$, $j = \overline{1,n}$, meeting

$$\lim_{t\to\infty} \boldsymbol{p}^k(t) = \boldsymbol{0},\ \ k = \overline{1,2}\,. \tag{6}$$

If for any $t_0 \in \Re_+$ and any initial condition $\boldsymbol{x}_0 = \left[\boldsymbol{x}_0^{1\tau}\ \boldsymbol{x}_0^{2\tau}\right]^\tau$, $x_0^1 \in \Re^m$, $x_0^2 \in \Re^n$, satisfying $\left|\boldsymbol{x}_0^k - \boldsymbol{x}_e^k\right| \le \boldsymbol{p}^k(t_0)$, $k = \overline{1,2}$, the corresponding solution to (1), $\boldsymbol{x}(t) = \left[\boldsymbol{x}^1(t)^\tau\ \boldsymbol{x}^2(t)^\tau\right]^\tau$, $\boldsymbol{x}^k(t) = \boldsymbol{x}^k(t; t_0, \boldsymbol{x}_0)$, $k = \overline{1,2}$, meets the inequality $\left|\boldsymbol{x}^k(t) - \boldsymbol{x}_e^k\right| \le \boldsymbol{p}^k(t)$, $\forall t \in \Re_+$, $t \ge t_0$, $k = \overline{1,2}$, then we say that the equilibrium point $\boldsymbol{x}_e$ of BAM (1) is *componentwise asymptotically stable* with respect to $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$, abbreviated as CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$).

(**b**) The equilibrium point $\boldsymbol{x}_e$ of BAM (1) is *globally* CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), or CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) *in the large*, abbreviated as GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), if $\boldsymbol{x}_e$ is CWAS($c\boldsymbol{p}^1, c\boldsymbol{p}^2$) for any scalar $c > 0$.

(**c**) BAM (1) is said to be CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) if it has an equilibrium point $\boldsymbol{x}_e$ that is GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$). ■

Remark 1. It can be proved that (**a**) if an equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ of BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), then it is also *uniformly asymptotically stable* in the sense of the standard definition (e.g. [14], pp. 107); (**b**) if an equilibrium point

$\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ of BAM (1) is GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), then it is also *uniformly asymptotically stable in the large* in the sense of the standard definition (e.g. [14], pp. 108). ■

Until this point of our presentation the time-dependence of the vector functions $\boldsymbol{p}^k(t)$, $k = \overline{1,2}$, was considered arbitrary. If the CWAS property exists for the particular form of the vector functions

$$\begin{aligned} &p^1(t) = e^{\sigma t}\alpha^1, p^2(t) = e^{\sigma t}\alpha^2, t \in \Re_+, \\ &\sigma \in (-\infty.0), \alpha^1 \in \Re^m, \alpha^1 > 0, \alpha^2 \in \Re^n, \alpha^2 > 0, \end{aligned} \tag{7}$$

then we refer to a special type of stability property called *componentwise exponential asymptotic stability*, abbreviated as CWEAS, and Definition 1 yields the following:

Definition 2. If the hypotheses of Definition 1(a, b, c) are fulfilled with $\boldsymbol{p}^k(t)$, $k = \overline{1,2}$, given by (7), then we say that: (**a**) the equilibrium point $\boldsymbol{x}_e$ is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$); (**b**) the equilibrium point $\boldsymbol{x}_e$ is globally CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), abbreviated as GCWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$); (**c**) BAM (1) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$). ■

Remark 2. It can be proved that (**a**) if an equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ of BAM (1) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), then it is also *exponentially asymptotically stable* in the classical sense (e.g. [14], pp. 107); (**b**) if an equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ of BAM (1) is GCWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$), then it is also *globally exponentially asymptotically stable* in the classical sense (e.g. [14], pp. 108). ■

3 Main Results

3.1 CWAS of BAMs with Uncertainties

Theorem 1. BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) for arbitrary $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k = \overline{1,2}$, if the following inequalities hold

$$\begin{aligned} \dot{\boldsymbol{p}}^1(t) &\geq \overline{\boldsymbol{A}}^1 \boldsymbol{p}^1(t) + \hat{\boldsymbol{W}}^2 \Lambda^2 \boldsymbol{p}^2(t), \\ \dot{\boldsymbol{p}}^2(t) &\geq \overline{\boldsymbol{A}}^2 \boldsymbol{p}^2(t) + \hat{\boldsymbol{W}}^1 \Lambda^1 \boldsymbol{p}^1(t), \end{aligned} \quad \forall t \in \Re_+, \tag{8}$$

where matrices $\hat{\boldsymbol{W}}^k$, Λ^k, $k = \overline{1,2}$, are defined by

$$
\begin{aligned}
&\hat{W}^1 = \left[\hat{w}_{ji}^1\right] \in \Re^{nxm}, \hat{w}_{ji}^1 = \max\left\{|\underline{w}_{ji}^1|, |\overline{w}_{ji}^1|\right\}, \\
&\hat{W}^2 = \left[\hat{w}_{ji}^2\right] \in \Re^{mxn}, \hat{w}_{ji}^2 = \max\left\{|\underline{w}_{ji}^2|, |\overline{w}_{ji}^2|\right\}, \\
&\Lambda^1 = diag\left\{\lambda_1^1, \lambda_2^1, ..., \lambda_m^1\right\} \in \Re^{mxm}, \\
&\Lambda^2 = diag\left\{\lambda_1^2, \lambda_2^2, ..., \lambda_m^2\right\} \in \Re^{nxn}.
\end{aligned} \tag{9}
$$

Proof: Given arbitrary $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k = \overline{1,2}$, the dynamical behavior of the state-space trajectories of BAM (1) in a vicinity of the equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\ \boldsymbol{x}_e^{2\tau}\right]^\tau$ may be analyzed by means of the deviations $\boldsymbol{y}^k = \boldsymbol{x}^k - \boldsymbol{x}_e^k$, $k = \overline{1,2}$, that satisfy

$$
\begin{aligned}
&\dot{\boldsymbol{y}}^1(t) = \boldsymbol{A}^1 \boldsymbol{y}^1(t) + \boldsymbol{W}^2 \boldsymbol{g}^2\left(\boldsymbol{y}^2(t)\right), \quad \boldsymbol{y}^1(t_0) = \boldsymbol{x}_0^1 - \boldsymbol{x}_e^1, \\
&\dot{\boldsymbol{y}}^2(t) = \boldsymbol{A}^2 \boldsymbol{y}^2(t) + \boldsymbol{W}^1 \boldsymbol{g}^1\left(\boldsymbol{y}^1(t)\right), \quad \boldsymbol{y}^2(t_0) = \boldsymbol{x}_0^2 - \boldsymbol{x}_e^2,
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
&\boldsymbol{g}^1(\boldsymbol{y}^1) = \boldsymbol{f}^1(\boldsymbol{y}^1 + \boldsymbol{x}_e^1) - \boldsymbol{f}^1(\boldsymbol{x}_e^1), \\
&\boldsymbol{g}^2(\boldsymbol{y}^2) = \boldsymbol{f}^2(\boldsymbol{y}^2 + \boldsymbol{x}_e^2) - \boldsymbol{f}^2(\boldsymbol{x}_e^2).
\end{aligned} \tag{11}
$$

Obviously, $\boldsymbol{x}_e$ is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) for (1) if and only if $\boldsymbol{y}_e = \boldsymbol{0}$ is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) for (10). The components g_i^1, $i = \overline{1,m}$, and g_j^2, $j = \overline{1,n}$, of the activation functions $\boldsymbol{g}^1$ and, respectively, $\boldsymbol{g}^2$, are continuous, nondecreasing and satisfy the *sector conditions* derived from (2):

$$
0 \le \frac{g_i^1(r)}{r} \le L_i^1, \quad 0 \le \frac{g_j^2(r)}{r} \le L_j^2, \tag{12}
$$

for all $r \in \Re, r \neq 0$.

For any $c > 0$, using equations (10) corresponding to a generic equilibrium point $\boldsymbol{x}_e$ of (1) and taking (12) into account, we get

$$
\begin{aligned}
& a_i^1 cp_i^1(t) - \sum_{j=1}^{n} w_{ij}^2 g_j^2(y_j^2) \le \overline{a}_i^1 cp_i^1(t) + \\
&\qquad + \sum_{j=1}^{n} \max\left\{ -\overline{w}_{ij}^2 g_j^2(-cp_j^2(t)), -\underline{w}_{ij}^2 g_j^2(cp_j^2(t)) \right\} \le \\
&\qquad \le \overline{a}_i^1 cp_i^1(t) + c\sum_{j=1}^{n} \lambda_j^2 \max\left\{ |\underline{w}_{ij}^2|, |\overline{w}_{ij}^2| \right\} p_j^2(t), \\
& a_i^1 cp_i^1(t) + \sum_{j=1}^{n} w_{ij}^2 g_j^2(y_j^2) \le \overline{a}_i^1 cp_i^1(t) + \\
&\qquad + \sum_{j=1}^{n} \max\left\{ \underline{w}_{ij}^2 g_j^2(-cp_j^2(t)), \overline{w}_{ij}^2 g_j^2(cp_j^2(t)) \right\} \le \\
&\qquad \le \overline{a}_i^1 cp_i^1(t) + c\sum_{j=1}^{n} \lambda_j^2 \max\left\{ |\underline{w}_{ij}^2|, |\overline{w}_{ij}^2| \right\} p_j^2(t),
\end{aligned}
\tag{13^1}
$$

and, analogously,

$$
\begin{aligned}
& a_j^2 cp_j^2(t) - \sum_{i=1}^{m} w_{ji}^1 g_i^1(y_i^1) \le \\
&\qquad \le \overline{a}_j^2 cp_j^2(t) + c\sum_{i=1}^{m} \lambda_i^1 \max\left\{ |\underline{w}_{ji}^1|, |\overline{w}_{ji}^1| \right\} p_i^1(t), \\
& a_j^2 cp_j^2(t) + \sum_{i=1}^{m} w_{ji}^1 g_i^1(y_i^1) \le \\
&\qquad \le \overline{a}_j^2 cp_j^2(t) + c\sum_{i=1}^{m} \lambda_i^1 \max\left\{ |\underline{w}_{ji}^1|, |\overline{w}_{ji}^1| \right\} p_i^1(t),
\end{aligned}
\tag{13^2}
$$

for all $t \ge 0$ and all $y_i^1 \in \left[-cp_i^1(t), cp_i^1(t)\right]$, $i = \overline{1,m}$, $y_j^2 \in \left[-cp_j^2(t), cp_j^2(t)\right]$, $j = \overline{1,n}$. If (8) is satisfied, it follows that

$$
\begin{aligned}
& \left.\begin{aligned}
& -c\dot{p}_i^1(t) \le -a_i^1 cp_i^1(t) + \sum_{j=1}^{n} w_{ij}^2 g_j^2(y_j^2), \\
& c\dot{p}_i^1(t) \ge a_i^1 cp_i^1(t) + \sum_{j=1}^{n} w_{ij}^2 g_j^2(y_j^2),
\end{aligned}\right\} \quad i = \overline{1,m}, \\
& \left.\begin{aligned}
& -c\dot{p}_j^2(t) \le -a_j^2 cp_j^2(t) + \sum_{i=1}^{m} w_{ji}^1 g_i^1(y_i^1), \\
& c\dot{p}_j^2(t) \le a_j^2 cp_j^2(t) + \sum_{i=1}^{m} w_{ji}^1 g_i^1(y_i^1),
\end{aligned}\right\} \quad j = \overline{1,n},
\end{aligned}
\tag{14}
$$

for all $t \ge 0$, and all $y_i^1 \in \left[-cp_i^1(t), cp_i^1(t)\right]$, $i = \overline{1,m}$, $y_j^2 \in \left[-cp_j^2(t), cp_j^2(t)\right]$, $j = \overline{1,n}$, which, according to [15] (Lemma 4.2, pp. 74) is sufficient for the equilibrium point $\boldsymbol{y}_e = \boldsymbol{0}$ of system (10) to be CWAS($c\boldsymbol{p}^1, c\boldsymbol{p}^2$). Since this happens for all

$c>0$, the equilibrium point $\boldsymbol{x}_e$ of BAM (1) is GCWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$), meaning that BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$). This conclusion can be drawn for all BAMs described by (1) with $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k=\overline{1,2}$, which completes the proof. ■

It is interesting to notice that the sufficient condition (8) stated by Theorem 1 can be equivalently written by using the augmented vector function

$$p:\Re_+ \to \Re^{m+n},\ \boldsymbol{p}(t)=\left[\boldsymbol{p}^1(t)^\tau \ \ \boldsymbol{p}^2(t)^\tau\right]^\tau, \tag{15}$$

and the matrix $\Theta \in \Re^{(m+n)x(m+n)}$ defined by

$$\Theta=\begin{bmatrix} \overline{\boldsymbol{A}}^1 & \boldsymbol{O} \\ \boldsymbol{O} & \overline{\boldsymbol{A}}^2 \end{bmatrix}+\begin{bmatrix} \boldsymbol{O} & \hat{\boldsymbol{W}}^2 \\ \hat{\boldsymbol{W}}^1 & \boldsymbol{O} \end{bmatrix}\begin{bmatrix} \Lambda^1 & \boldsymbol{O} \\ \boldsymbol{O} & \Lambda^2 \end{bmatrix}=\begin{bmatrix} \overline{\boldsymbol{A}}^1 & \hat{\boldsymbol{W}}^2\Lambda^2 \\ \hat{\boldsymbol{W}}^1\Lambda^1 & \overline{\boldsymbol{A}}^2 \end{bmatrix}, \tag{16}$$

with matrices $\hat{\boldsymbol{W}}^k$, Λ^k, $k=\overline{1,2}$, given by (9).

Corollary 1. BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) for arbitrary $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k=\overline{1,2}$, if the vector function $\boldsymbol{p}(t)$ (15) satisfies the differential inequality

$$\dot{p}(t) \geq \Theta p(t), \forall t \in \Re_+, \tag{17}$$

where matrix Θ is given by (16). ■

Corollary 1 suggests us to explore the role of matrix Θ in ensuring the CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) of BAM (1) with uncertainties. Let us first notice the special structure of the test matrix Θ given by (16), which is essentially nonnegative (all its off-diagonal elements are nonnegative). This remark motivates us to present some preparatory results.

Lemma 1. Let $\Psi=\left[\Psi_{ij}\right] \in \Re^{qxq}$ be an essentially nonnegative matrix and let us denote by $\upsilon_i(\Psi)$, $i=\overline{1,q}$, its eigenvalues. Then, Ψ has a real eigenvalue (simple or multiple), denoted by $\upsilon_{\max}(\Psi)$, which fulfills the dominance condition $\mathrm{Re}[\upsilon_i(\Psi)] \leq \upsilon_{\max}(\Psi)$ for all $i=\overline{1,q}$. Moreover, $\psi_{ii} \leq \upsilon_{\max}(\Psi)$, $i=\overline{1,q}$.

Proof: It results from ([8], Lemma 2.1) and from ([16], Corollary 8.1.20). ■

Lemma 2: If both matrices $\Psi, \Xi \in \Re^{qxq}$ are essentially nonnegative matrices with $\Psi \leq \Xi$, then $\upsilon_{\max}(\Psi) \leq \upsilon_{\max}(\Xi)$.

Proof: Consider $s>0$ such that $\psi_{ii}+s\geq 0$, $i=\overline{1,q}$. Thus, the matrices $\boldsymbol{\Psi}+s\boldsymbol{I}\leq \Xi+s\boldsymbol{I}$ are nonnegative and we use ([16], Theorem 8.1.18). ■

Lemma 3: If $\Psi=\left[\Psi_{ij}\right]\in\Re^{qxq}$ is an essentially nonnegative matrix, then, for any $r\in\Re$, with $\upsilon_{\max}(\Psi)<r$, there exists a positive vector $\gamma\in\Re^{n},\gamma>0$, such that $\boldsymbol{\Psi}\gamma<r\gamma$.

Proof: Construct the nonnegative matrix $s\boldsymbol{I}+\boldsymbol{\Psi}$ with $s+\psi_{ii}\geq 0$, $i=\overline{1,q}$. For any $r\in\Re$ satisfying $\upsilon_{\max}(\Psi)<r$, there exists $\varepsilon=\varepsilon(r)>0$ such that $\upsilon_{\max}(s\boldsymbol{I}+\boldsymbol{\Psi}+\varepsilon\boldsymbol{E})\leq s+r$, where $E=\left[e_{ij}\right]\in\Re^{qxq}$, with $e_{ij}=1$, $i,j=\overline{1,q}$. Thus, for the Perron eigenvector $\gamma\in\Re^{q},\gamma>0$, of the positive matrix $s\boldsymbol{I}+\boldsymbol{\Psi}+\varepsilon\boldsymbol{E}>\boldsymbol{0}$ we can write

$$(s\boldsymbol{I}+\boldsymbol{\Psi})\boldsymbol{\gamma}<(s\boldsymbol{I}+\boldsymbol{\Psi}+\varepsilon\boldsymbol{E})\boldsymbol{\gamma}=\upsilon_{\max}(s\boldsymbol{I}+\boldsymbol{\Psi}+\varepsilon\boldsymbol{E})\boldsymbol{\gamma}\leq(s+r)\boldsymbol{\gamma}\ .$$

Note that when $\boldsymbol{\Psi}$ is irreducible, the existence of its Perron-Frobenius eigenvector $\gamma\in\Re^{q},\gamma>0$, ensures the equality $\boldsymbol{\Psi}\gamma=\upsilon_{\max}(\boldsymbol{\Psi})\gamma$. ■

We are now able to establish the following result.

Theorem 2. If matrix $\boldsymbol{\Theta}$ defined by (16) is Hurwitz stable, then there exist two vector functions $\boldsymbol{p}^{k}(t)$, $k=\overline{1,2}$, satisfying the conditions from Definition 1 so that BAM (1) is CWAS($\boldsymbol{p}^{1},\boldsymbol{p}^{2}$) for all $\boldsymbol{A}^{k}\in\mathcal{A}^{k}$, $\boldsymbol{W}^{k}\in\mathcal{W}^{k}$, $\boldsymbol{f}^{k}\in\mathcal{F}^{k}$, $k=\overline{1,2}$.

Proof: Indeed, if $\boldsymbol{\Theta}$ is Hurwitz stable, then the vector function

$$\boldsymbol{p}(t)=\mathrm{e}^{\boldsymbol{\Theta}t}\,\boldsymbol{p}(0)+\int_{0}^{t}\mathrm{e}^{\boldsymbol{\Theta}(t-\xi)}\,\boldsymbol{v}(\xi)d\xi\ ,\ t>0\ ,$$

(defined with $\boldsymbol{p}(0)>\boldsymbol{0}$ and adequate $\boldsymbol{v}(\xi)\geq 0$, $\xi\geq 0$, such that $\lim_{t\to\infty}\int_{0}^{t}\mathrm{e}^{\boldsymbol{\Theta}(t-\xi)}\,\boldsymbol{v}(\xi)d\xi=\boldsymbol{0}$), satisfies the differential inequality (17) and $\boldsymbol{p}(t)>\boldsymbol{0}$, $\forall t\in\Re_{+}$, $\lim_{t\to\infty}\boldsymbol{p}(t)=\boldsymbol{0}$. The two functions ensuring CWAS($\boldsymbol{p}^{1},\boldsymbol{p}^{2}$) result from the appropriate partitioning of $\boldsymbol{p}$, in accordance to (15). ■

Remark 3. Since matrix $\boldsymbol{\Theta}$ defined by (16) is essentially nonnegative, the requirement "$\boldsymbol{\Theta}$ Hurwitz stable" from the hypothesis of Theorem 2 is equivalent to "$-\boldsymbol{\Theta}$ nonsingular M-matrix" [18]. ■

3.2 CWEAS of BAMs with Uncertainties

Theorem 3. BAM (1) is CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$) for all $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k=\overline{1,2}$, if the following algebraic inequalities hold

$$\begin{aligned} \sigma\boldsymbol{\alpha}^1 &\geq \overline{\boldsymbol{A}}^1\boldsymbol{\alpha}^1 + \hat{\boldsymbol{W}}^2\Lambda^2\boldsymbol{\alpha}^2, \\ \sigma\boldsymbol{\alpha}^2 &\geq \overline{\boldsymbol{A}}^2\boldsymbol{\alpha}^2 + \hat{\boldsymbol{W}}^1\Lambda^1\boldsymbol{\alpha}^1. \end{aligned} \tag{18}$$

Proof: It is a direct consequence of Theorem 1 when the time-dependence of $\boldsymbol{p}^k(t)$, $k=\overline{1,2}$, is given by (7). ■

Corollary 2. BAM (1) is CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$) for all $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k=\overline{1,2}$, if the augmented vector $\alpha = \left[\alpha^{1\tau}\alpha^{2\tau}\right]^\tau \in \Re^{m+n}$ satisfies the following algebraic inequality

$$\sigma\boldsymbol{\alpha} \geq \Theta\boldsymbol{\alpha}, \tag{19}$$

where Θ is given by (16). ■

Similarly to Theorem 2, the following result is available for CWEAS.

Theorem 4. If matrix Θ defined by (16) is Hurwitz stable, then there exist two positive vectors $\alpha^1 \in \Re^m, \alpha^1 > 0$, $\alpha^2 \in \Re^n, \alpha^2 > 0$, and a scalar $\sigma \in (-\infty,0)$, so that BAM (1) is CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$) for all $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k=\overline{1,2}$.

Proof: If matrix Θ is Hurwitz stable, then Lemma 3 ensures the existence of $\sigma \in \Re$, $\upsilon_{\max}(\Theta) < \sigma < 0$ and $\alpha \in \Re^{m+n}, \alpha > 0$, satisfying inequality (19). The two positive vectors ensuring CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$) result from the appropriate partitioning of $\boldsymbol{\alpha}$. ■

Remark 4. If BAM (1) is CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$) for all $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k=\overline{1,2}$, then, for the unique equilibrium point $\boldsymbol{x}_e = \left[\boldsymbol{x}_e^{1\tau}\,\boldsymbol{x}_e^{2\tau}\right]^\tau$ of each concrete neural network belonging to this family, we can write

$$\begin{aligned} &\forall \varepsilon > 0, \forall t_0 \in \Re_+, \forall x_0 \in \Re^n, \\ &\| x_0 - x_e \|_\infty^A \leq \varepsilon \Rightarrow \| x(t;t_0,x_0) - x_e \|_\infty^A \leq \varepsilon e^{r(t-t_0)}, \forall t \in \Re_+, t \geq t_0 \end{aligned}, \tag{20}$$

where the vector norm $\| \|_\infty^{\mathrm{A}}$ is defined by $\|\boldsymbol{x}\|_\infty^{\mathrm{A}} = \|\mathrm{A}\boldsymbol{x}\|_\infty$, $\forall x \in \Re^{m+n}$, with

$$\mathrm{A} = \mathrm{diag}\left\{1/\alpha_1^1, 1/\alpha_2^1, \ldots, 1/\alpha_m^1, 1/\alpha_1^2, 1/\alpha_2^2, \ldots, 1/\alpha_n^2\right\}. \tag{21}$$

This shows that for each concrete neural network the definition of exponential stability of $\boldsymbol{x}_e$ (e.g. [14], pp.107) with respect to the norm $\| \; \|_\infty^{\mathrm{A}}$ is fulfilled in the particular case $\delta(\varepsilon)=\varepsilon$, $\forall \varepsilon>0$. Similarly, the definition of global exponential stability of $\boldsymbol{x}_e$ (e.g. [14], pp.108) is satisfied for the particular case $M=1$,

$$\forall t_0 \in \Re_+, \forall x_0 \in \Re^n,$$
$$\| x(t;t_0,x_0)-x_e \|_\infty^A \le M e^{r(t-t_0)} \| x_0 - x_e \|_\infty^A, \forall t \ge t_0 \tag{22}$$

Remark 5. In terms of the matrix norms induced by the vector norms, the sufficient condition for BAM (1) to be CWEAS($\sigma,\boldsymbol{\alpha}^1,\boldsymbol{\alpha}^2$) for all $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $\boldsymbol{f}^k \in \mathcal{F}^k$, $k=\overline{1,2}$, can be formulated as

$$\mu_{\| \|_\infty^{\mathrm{A}}}(\Theta)=\mu_{\| \|_\infty}\left(\mathrm{A}\Theta\mathrm{A}^{-1}\right)\le r<0, \tag{23}$$

where matrix A is given by (21) and $\mu_{\| \|}(\boldsymbol{M})=\lim_{\varsigma\downarrow 0}(\|\boldsymbol{I}+\varsigma\boldsymbol{M}\|-1)/\varsigma$ defines a measure for any matrix $M\in\Re^{nxn}$, associated with the induced matrix norm $\| \; \|$ (e.g. [17], pp. 30). This is a direct consequence of Corollary 2. ■

4 Particular Cases

The generality of our results on CWAS / CWEAS of BAM with uncertainties includes some particular cases, which deserve a special interest and allow meaningful comparisons with the works of other authors.

4.1 CWAS / CWEAS of BAMs with Activation-function Uncertainties

This case is obtained when $\underline{\boldsymbol{A}}^k=\overline{\boldsymbol{A}}^k=\boldsymbol{A}^k$, $\underline{\boldsymbol{W}}^k=\overline{\boldsymbol{W}}^k=\boldsymbol{W}^k$, $k=\overline{1,2}$, in (3) and the uncertainties refer only to the class of activation functions defined by (5). The CWAS/CWEAS approach relies on the replacement of the test matrix $\Theta\in\Re^{(m+n)x(m+n)}$, built according to (16), by the simplified test matrix $\Omega\in\Re^{(m+n)x(m+n)}$ defined as:

$$\Omega = \begin{bmatrix} \boldsymbol{A}^1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}^2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{O} & |\boldsymbol{W}^2| \\ |\boldsymbol{W}^1| & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \Lambda^1 & \boldsymbol{O} \\ \boldsymbol{O} & \Lambda^2 \end{bmatrix} = \\ = \begin{bmatrix} \boldsymbol{A}^1 & |\boldsymbol{W}^2|\Lambda^2 \\ |\boldsymbol{W}^1|\Lambda^1 & \boldsymbol{A}^2 \end{bmatrix}, \tag{24}$$

with matrices Λ^k, $k = \overline{1,2}$, given by (9).

Theorem 5. If matrix $\boldsymbol{\Omega}$ defined by (24) is Hurwitz stable, then

(**a**) BAM (1) is CWAS($\boldsymbol{p}^1, \boldsymbol{p}^2$) for all $\boldsymbol{f}^k \in \mathcal{F}^k$, $k = \overline{1,2}$, if $\boldsymbol{p}(t)$ given by (15) fulfills the differential inequality

$$\dot{\boldsymbol{p}}(t) \geq \boldsymbol{\Omega p}(t) ;\ \forall t \in \Re_+ ; \tag{25}$$

(**b**) BAM (1) is CWEAS($\sigma, \boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2$) for all $\boldsymbol{f}^k \in \mathcal{F}^k$, $k = \overline{1,2}$, if the scalar σ and the vector $\alpha = \left[\alpha^{1\tau} \alpha^{2\tau}\right]^\tau \in \Re^{m+n}$ fulfill the algebraic inequality

$$\sigma\boldsymbol{\alpha} \geq \boldsymbol{\Omega\alpha} . \quad \blacksquare \tag{26}$$

Remark 6. The conditions $a_i^1 < 0$, $i = \overline{1,m}$, $a_j^2 < 0$, $j = \overline{1,n}$, are necessary for matrix $\boldsymbol{\Omega}$ defined by (24) to be Hurwitz stable. Such conditions are formulated as working hypotheses in most papers dealing with BAMs. ■

4.2 CWAS / CWEAS of BAMs with Parameter Uncertainties

This case is obtained when the activation functions f_i^1, $i = \overline{1,m}$, and f_j^2, $j = \overline{1,n}$, are fixed, satisfying (2), and the uncertainties refer only to the classes of matrices $\mathcal{A}^k$, $\mathcal{W}^k$, $k = \overline{1,2}$, defined by (4). The CWAS/CWEAS approach relies on the test matrix $\boldsymbol{\Theta}$ defined by (16), constructed with $\Lambda^1 = \mathrm{diag}\{L_1^1, L_2^1, \ldots, L_m^1\}$ and $\Lambda^2 = \mathrm{diag}\{L_1^2, L_2^2, \ldots, L_n^2\}$, where $L_i^1, L_j^2 > 0$ are the Lipschitz constants corresponding to the activation functions, as shown by (2).

Theorems 2 and 4 provide sufficient conditions for BAM (1) to be CWAS / CWEAS for all $\boldsymbol{A}^k \in \mathcal{A}^k$, $\boldsymbol{W}^k \in \mathcal{W}^k$, $k = \overline{1,2}$.

4.3 CWAS / CWEAS of BAMs without Uncertainties

This case is obtained from cases *A* and *B* discussed above, when $\underline{\boldsymbol{A}}^k = \overline{\boldsymbol{A}}^k = \boldsymbol{A}^k$, $\underline{\boldsymbol{W}}^k = \overline{\boldsymbol{W}}^k = \boldsymbol{W}^k$, $k = \overline{1,2}$, in (3), and the activation functions f_i^1, $i = \overline{1,m}$, and f_j^2, $j = \overline{1,n}$, are fixed, satisfying (2). The CWAS/CWEAS approach relies on the test matrix Ω defined by (24), with $\Lambda^1 = \text{diag}\{L_1^1, L_2^1, \ldots, L_m^1\}$ and $\Lambda^2 = \text{diag}\{L_1^2, L_2^2, \ldots, L_n^2\}$, where $L_i^1, L_j^2 > 0$ are the Lipschitz constants from (2).

Thus, Theorem 5 provides a sufficient condition for the componentwise stability of BAM (1), namely "Ω is Hurwitz stable", or, equivalently, "-Ω is an M-matrix" This result can be compared with others reported in literature for BAMs. For instance, starting from the same condition on Ω and applying Theorem 3.1 in [2] when delayed states do not occur, only the global asymptotic stability of the BAM is guaranteed. A similar comment may be formulated with respect to Theorem 1 in [3] that, for a BAM without delay, ensures only the global exponential stability.

Those results in [1] dealing with BAMs without delay deserve a special attention. Corollaries 2.2 and 2.4 from [1] rely on the hypothesis of diagonal dominance for matrix Ω (applied on columns and, respectively, on rows), which means a stronger test condition than the Hurwitz stability of matrix Ω. Under these hypotheses, the global exponential stability of a BAM is guaranteed by Corollary 2.2 in the sense of Hölder norm 1 and by Corollary 2.4 in the sense of Hölder norm ∞. The latter result represents a particularization of our CWEAS framework obtained for $\alpha_i^1 = \alpha_j^2 = \alpha$, $i = \overline{1,m}$, $j = \overline{1,n}$, in (20) and (21).

Conclusions

This paper provides easy-to-apply algebraic criteria for exploring the componentwise (exponential) asymptotic stability of BAM neural networks with parameter and activation-function uncertainties. These criteria are formulated in terms of Hurwitz stability of a test matrix adequately built from the information available about the uncertainties. Besides the generality of these novel robustness results (Theorems 2 and 4), they allow deriving some relevant particular cases for the analysis of CWAS / CWEAS. Comparisons between the mentioned particular cases and other works reveal that, for frequently encountered systems (i.e. with nondecreasing activation functions), our test condition is less restrictive and, concomitantly, ensures stronger stability properties.

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