On DMV-algebras and Product MV-algebras

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Abstract. Alternate definitions of Double Product MV-algebras and Product MV-algebras, that are equivalent with the original ones are introduced. Double Product MV-algebras and Product MV-algebras are similar algebraic structures, but not equivalent. Their relationship is investigated and an example is given.

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1 Introduction

MV-algebras were introduced in 1958 by Chang [1] and since then this structure has captured the interest of many mathematicians. An equivalent definition of MV-algebras is given by Mundici [2], as follows:

Definition 1.1 An MV-algebra is an algebra $(M, \oplus, \neg, 0_M)$ with a binary operation $\oplus$, a unary operation $\neg$ and a constant 0_M satisfying the following equations:

i) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

ii) $x \oplus y = y \oplus x$;

iii) $x \oplus 0_M = x$;

iv) $\neg\neg x = x$;

v) $x \oplus \neg 0_M = \neg 0_M$;

vi) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Remark 1.2 The constant 1_M and the operations $\odot$ and $\ominus$ are defined on each MV-algebra M as it follows:

i) $1_M = \neg 0_M$;

ii) $x \odot y = \neg(\neg x \oplus \neg y)$;

iii) $x \ominus y = x \odot \neg y$.

On this structure there is defined a partial order relation as follows:

Definition 1.3 Let M be an MV-algebra and $x, y \in M$. We say that $x \leq y$ if and only if x and y satisfy one of the bellow equivalent conditions:

i) $\neg x \oplus y = 1_M$;

ii) $x \odot \neg y = 0_M$;

iii) $y = x \oplus (y \ominus x)$;

iv) there is an element $z \in M$ such that $x \oplus z = y$.

Mundici [6] proved that there exists a categorical equivalence between the concept of MV-algebras and (Abelian) l-groups with strong unity. This means that any MV-algebra may be obtained from an l-group G with strong unit.

In what follows we indicate the way of obtaining an MV-algebra from an l-group G with strong unit.

Definition 1.4 Let $(G, +, 0, \leq)$ be an Abelian l-group. We say that $u \in G$ is a *strong unit* of G if for any $v \in G$ there is an integer $n \geq 1$ such that $-nu \leq v \leq nu$.

Let now us consider the following operations:

i) $\oplus : [0,u] \times [0,u] \to [0,u]$ where for any $x, y \in [0,u]$ we have

$x \oplus y = (x + y) \wedge u$;

ii) $\neg : [0,u] \to [0,u]$ where for any $x \in [0,u]$ we have

$\neg x = u - x$.

Theorem 1.5 The structure $([0,u],\oplus,\neg,0)$ is an MV-algebra.

The concept of Double Product MV-algebra has been originally introduced by Dumitrescu in [3] (see also [4]) and has been studied in [5]. Double Product MV-algebra enriches MV-algebra with a binary internal operation called *product*. The operation $\oplus$ induces in an MV-algebra the product $\odot$. Therefore the extra operation $\cdot$ of Double Product MV-algebras may be considered as a supplementary product. This is the reason of choosing the name Double Product MV-algebras.

Definition 1.6 [5] A Double Product MV-algebra (shortly DMV) is an algebraic structure $(M,\oplus,\cdot,\neg,0_M)$ fulfilling the following axioms:

i) $(M,\oplus,\neg,0_M)$ is an MV-algebra;

ii) $(M,\cdot)$ is a semigroup;

iii) if $a \odot b = 0_M$ then the equalities

$$c\cdot(a\oplus b) = c\cdot a \oplus c\cdot b,$$

and

$$(a\oplus b)\cdot c = a\cdot c \oplus b\cdot c,$$

hold for any $a,b,c\in M$.

Remark 1.7 As we can see the new product $\cdot$ is considered to be distributive with respect to the operation $\oplus$ defined in the MV-algebra $(M,\oplus,\neg,0_M)$.

Di Nola and Dvurecenskij [7] introduced Product MV-algebras also by enriching MV-algebras with a new internal binary operation called product.

Definition 1.8 [7] A Product MV-algebra (shortly PMV) is an algebraic structure $(M,\oplus,\cdot,\neg,0_M)$ fulfilling the following axioms:

i) $(M,\oplus,\neg,0_M)$ is an MV-algebra;

ii) $a\cdot(b\cdot c) = (a\cdot b)\cdot c$;

iii) if $a+b$ is defined in M, then $a\cdot c + b\cdot c$ and $c\cdot a + c\cdot b$ exists and the equalities

$$\left(a+b\right)\cdot c = a\cdot c + b\cdot c,$$

and

$$c\cdot\left(a+b\right) = c\cdot a + c\cdot b,$$

hold for any $a,b,c\in M$.

Remark 1.9 It is important to notice that the new product is distributive according to the partial binary operation $+$ induced on M by the binary operation from the l-group G that generates the MV-algebra $\left(M,\oplus,\neg,0_M\right)$.

It was also proved [7] that there exists a categorical equivalence between the category of associative l-rings with a strong unit u such that $u\cdot u\le u$ and the category of Product MV-algebras.

Even that the definitions of DMV and PMV look quite similar, as we will see later these definitions are not equivalent.

In what follows we denote the class of DMV by C_{DMV} and the class of PMV by C_{PMV}.

2 New Definitions of DMV and PMV

Since the definitions of DMV and PMV look quite similar, we introduce an alternate definition for DMV, that will allow us to establish the relationship between these two algebraic structures. We prove that the alternate definition is equivalent with the original one.

Definition 2.1 A Double Product MV-algebra (shortly DMV) is an algebraic structure $\left(M,\oplus,\cdot,\neg,0_M\right)$ fulfilling the following axioms:

i) $\left(M,\oplus,\neg,0_M\right)$ is an MV-algebra;

ii) $a\cdot\left(b\cdot c\right)=\left(a\cdot b\right)\cdot c$;

iii) if $a+b$ is defined in M, then the equalities

$$\left(a+b\right)\cdot c = a\cdot c\oplus b\cdot c,$$

and

$$c \cdot (a+b) = c \cdot a \oplus c \cdot b,$$

hold for any $a, b, c \in M$.

Remark 2.2 As we can see the alternate definition of the DMV is more similar with the definition of PMV than the original one.

The question that occurs, is if the alternate definition is equivalent with the original one. The answer is supplied by the next Theorem.

Theorem 2.3 Definitions 1.6 and 2.1 of Double Product MV-algebras are equivalent.

Proof. In both definitions we start from the same algebraic structure $(M, \oplus, \cdot, \neg, 0_M)$. In what follows we have to check if the axioms are equivalent.

The second axiom of Definition 1.6 states that $(M, \cdot)$ is a subgroup, that is equivalent with

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c,$$

which is the second axiom of Definition 2.1.

Axiom iii) of Definition 2.1 involves the condition:

$a + b$ is defined.

This condition is equivalent to

$$a + b \leq 1_M.$$

The last inequality is also equivalent with

$$a \leq 1_M - b$$

$$= \neg b.$$

From axiom ii) of Definition 1.3, we obtain that the condition:

$$a + b \leq 1_M,$$

is equivalent with

$$a \odot b = 0_M.$$

But the previous equality is just the condition from axiom iii) of Definition 1.6.

It means that the two conditions are equivalent. We have now only to compare the distributivity conditions that appear in the both definitions.

Equality

$$c \cdot (a \oplus b) = c \cdot a \oplus c \cdot b$$

is axiom iii) of Definition 1.6 and is equivalent to

$$c \cdot \left((a + b) \wedge 1_M \right) = c \cdot a \oplus c \cdot b.$$

We proved that

$$a \odot b = 0_M \Longleftrightarrow a + b \text{ is defined.}$$

Since

$$a + b \leq 1_M$$

if follows that the previous equality is equivalent to

$$c \cdot (a + b) = c \cdot a \oplus c \cdot b,$$

expressing the distributivity condition from Definition 1.

It follows that the third axioms of the two definitions are equivalent and this completes the proof. $\square$

We are ready now to introduce an alternate definition for PMV :

Definition 2.4 A Product MV-algebra (shortly PMV) is an algebraic structure $(M, \oplus, \cdot, \neg, 0_M)$ fulfilling the following axioms:

i) $(M, \oplus, \neg, 0_M)$ is an MV-algebra;

ii) $(M, \cdot)$ is a semigroup;

iii) if $a \odot b = 0_M$ then the following equalities:

$$(c \cdot a) \odot (c \cdot b) = 0_M, a \tag{1}$$

$$(a \cdot c) \odot (b \cdot c) = 0_M, b \tag{2}$$

$$c \cdot (a + b) = c \cdot a + c \cdot b, c \tag{3}$$

$$(a + b) \cdot c = a \cdot c + b \cdot c, d \tag{4}$$

hold for any $a, b, c \in M$.

Theorem 2.5 The tow definitions of a Product MV-algebras are equivalent.

Proof. The proof is quite similar to the proof of Theorem 2.3. □

3 Relationship between DMV and PMV

The relationship between Double Product MV-algebras and Product MV-algebras is investigated. The main question is if one structure may be viewed as a particular case of the other one.

In the process of establishing which algebraic structure is more general, Definition 2.1 of DMV and Definition 1.8 of PMV are considered.

The first two axioms of the considered definitions are identical. It means that the difference between the two algebraic structures derives from the third axiom.

In the definition of PMV the distributivity is introduced by using the partial binary operation $+$ induced on M by the binary operation $+$ from the l-group that generates the MV-algebra M as follows:

$$(a+b)\cdot c = a\cdot c + b\cdot c. \tag{5}$$

Since $a\cdot c + b\cdot c$ exists (Definition 1.8, iii)) it means that

$$a\cdot c + b\cdot c \le 1_M.$$

It follows that

$$a\cdot c \oplus b\cdot c = (a\cdot c + b\cdot c)\wedge 1_M \tag{6}$$

$$= a\cdot c + b\cdot c.$$

From equations (5) and (6) we obtain

$$(a+b)\cdot c = a\cdot c \oplus b\cdot c,$$

which is the distributivity condition from the definition of DMV (Definition 2.1, iii)).

In axiom iii) of Definition 1.8 we have the equality

$$c\cdot(a+b) = c\cdot a + c\cdot b. \tag{7}$$

Since $c\cdot a + c\cdot b$ exists (Definition 1.8, iii)) it means that

$$c\cdot a + c\cdot b \le 1_M.$$

It follows that

$$c \cdot a + c \cdot b = (c \cdot a + c \cdot b) \wedge 1_M \tag{8}$$

$$= c \cdot a \oplus c \cdot b.$$

From equations (7) and (8) we obtain

$$c \cdot (a + b) = c \cdot a \oplus c \cdot b,$$

which is the second distributivity condition from the definition of DMV(Definition 2.1, iii)).

The above results lead us to the following Theorem:

Theorem 3.1 Any Product MV-algebra is a DMV-algebra.

Proof. The proof of this theorem is sustained by the results obtained above. □

Remark 3.2 The previous theorem states that the implication

$$P \in C_{PMV} \Rightarrow P \in C_{DMV},$$

holds.

In what follows we check if the reverse of implication also holds. We start from the definition of distributivity from Definition 2.1:

$$(a + b) \cdot c = a \cdot c \oplus b \cdot c \tag{9}$$

$$= (a \cdot c + b \cdot c) \wedge 1_M.$$

It follows that if

$$a \cdot c + b \cdot c \leq 1_M$$

we have

$$(a + b) \cdot c = a \cdot c + b \cdot c, \tag{10}$$

which is the definition of distributivity from Definition 1.8.

From equation (9) also follows that if

$$a \cdot c + b \cdot c > 1_M$$

the equation (10) does not hold.

The previous results lead us to the following Theorem:

Theorem 3.3 Not any DMV-algebra is an Product MV-algebra.

Remark 3.4 The previous theorem means that even if $D \in C_{DMV}$, there are situations when $D \notin C_{PMV}$.

Theorems 3.1 and 3.3 lead us to the main result of this paper:

Theorem 3.5 The class of Product MV-algebras is strictly included in the class of DMV-algebras.

Proof. From Theorem 3.1 we have that

$$C_{PMV} \subseteq C_{DMV}. \tag{11}$$

From Theorem 3.3 we also have that

$$C_{DMV} \not\subseteq C_{PMV}. \tag{12}$$

From (11) and (12) we have that

$$C_{PMV} \subset C_{DMV}.$$

This completes the proof. □

Remark 3.6 Since $C_{PMV} \subset C_{DMV}$ it follows that DMV-algebras is a more general structure than Product MV-algebras.

The above results are clearly establishing the relation between the Double Product MV-algebras and Product MV-algebras.

4 An Example of Structure that is an DMV and is not an PMV

In ([8]) it was proved that the structure $\left(\left[0, 2^t - 1\right], \oplus, \neg, 0\right)$ is an MV-algebra with the operations defined as follows:

$$a \oplus b = \left(a + b\right) \wedge \left(2^t - 1\right)$$

and

$$\neg a = 2^t - 1 - a.$$

Let $t = 2$. It follows that $\left(\left[0, 3\right], \oplus, \neg, 0\right)$ is an MV-algebra with the operations defined as follows:

$a \oplus b = (a+b) \wedge 3$

and

$\neg a = 3 - a.$

Let us now consider the binary multiplicative operation $\bullet : [0,3] \times [0,3] \to [0,3]$ defined as follows:

$a \bullet b = (a \cdot b) \wedge 3,$

where the binary operation $\cdot$ is the usual product of real numbers.

It is easy to prove that $([0,3], \bullet)$ is a subgroup and that the structure $([0,3], \oplus, \bullet, \neg, 0)$ is a Double Product MV-algebra.

Let $x = 1.4$, $y = 1.4$ and $z = 1.2$ be three numbers from $[0,3]$ interval. It is obvious that $x + y \leq 3.$

Let us assume that $([0,3], \oplus, \bullet, \neg, 0)$ is a Product MV-algebra. Since $x + y \leq 3$ it follows that

$$(x+y) \bullet z = x \bullet z + y \bullet z. \qquad (13)$$

But

$(x+y) \bullet z = (1.4 + 1.4) \bullet 1.2$

$= 2.8 \bullet 1.2$

$= 3.36 \wedge 3$

$= 3$

and

$x \bullet z + y \bullet z = 1.4 \bullet 1.2 + 1.4 \bullet 1.2$

$= 1.68 \wedge 3 + 1.68 \wedge 3$

$= 1.68 + 1.68$

$= 3.36.$

Since $3.36 \notin [0,3]$ it follows that $x \bullet z + y \bullet z$ is not defined in $[0,3]$ and it follows that equation (13) does not hold for these values of x, y and z. It means that the assumptiom that $\left(\left[0,3\right],\oplus,\bullet,\neg,0\right)$ is a Product MV-algebra is not correct.

It follows that even if the structure $\left(\left[0,3\right],\oplus,\bullet,\neg,0\right)$ is a Double product MV-algebra, it is not a Product MV-algebra.

This example shows that the inclusion in the Theorem 3.5 is strict.

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