Some illustrations of possibilistic correlation

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Abstract: In this paper we will show some examples for possibilistic correlation. In particular, we will show (i) how the possibilistic correlation coefficient of two linear marginal possibility distributions changes from zero to -1/2, and from -1/2 to -3/5 by taking out bigger and bigger parts from the level sets of a their joint possibility distribution; (ii) how to compute the autocorrelation coefficient of fuzzy time series with linear fuzzy data.

1 Introduction

A *fuzzy number* A is a fuzzy set $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by $\mathcal{F}$. Fuzzy numbers can be considered as possibility distributions. A fuzzy set C in $\mathbb{R}^2$ is said to be a joint possibility distribution of fuzzy numbers $A, B \in \mathcal{F}$, if it satisfies the relationships $\max\{x \mid C(x,y)\} = B(y)$ and $\max\{y \mid C(x,y)\} = A(x)$ for all $x, y \in \mathbb{R}$. Furthermore, A and B are called the marginal possibility distributions of C. Let $A \in \mathcal{F}$ be fuzzy number with a γ-level set denoted by $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$ and let U_γ denote a uniform probability distribution on $[A]^\gamma$, $\gamma \in [0,1]$.

In possibility theory we can use the principle of *expected value* of functions on fuzzy sets to define variance, covariance and correlation of possibility distributions. Namely, we equip each level set of a possibility distribution (represented by a fuzzy number) with a uniform probability distribution, then apply their standard probabilistic calculation, and then define measures on possibility distributions by integrating these weighted probabilistic notions over the set of all membership grades. These weights (or importances) can be given by weighting functions. A function $f\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monotone increasing and satisfies the following normalization condition $\int_0^1 f(\gamma) d\gamma = 1$. Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions.

In 2009 we introduced a new definition of possibilistic correlation coefficient (see [2]) that improves the earlier definition introduced by Carlsson, Fullér and Majlender in 2005 (see [1]).

Definition ([2])**.** *The f-weighted possibilistic correlation coefficient of $A, B \in \mathcal{F}$ (with respect to their joint distribution C) is defined by*

$$\rho_f(A,B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) \mathrm{d}\gamma \tag{1}$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}}$$

and, where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$, and $\mathrm{cov}(X_\gamma, Y_\gamma)$ denotes their probabilistic covariance.

In other words, the f-weighted possibilistic correlation coefficient is nothing else, but the f-weighted average of the probabilistic correlation coefficients $\rho(X_\gamma, Y_\gamma)$ for all $\gamma \in [0,1]$.

Consider the case, when $A(x) = B(x) = (1-x) \cdot \chi_{[0,1]}(x)$, for $x \in \mathbb{R}$, that is $[A]^\gamma = [B]^\gamma = [0, 1-\gamma]$, for $\gamma \in [0,1]$. Suppose that their joint possibility distribution is given by $F(x,y) = (1-x-y) \cdot \chi_T(x,y)$, where

$$T = \left\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\right\}.$$

Then we have $[F]^\gamma = \left\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1-\gamma\right\}$.

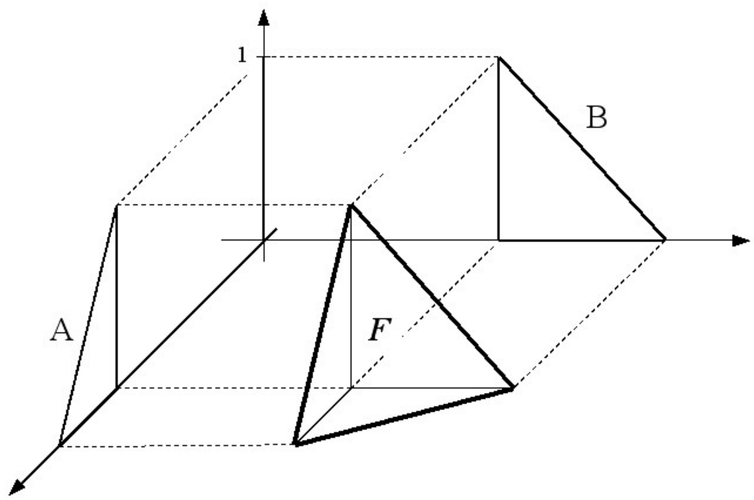

Figure 1: Illustration of joint possibility distribution F.

This situation is depicted on Fig. 1, where we have shifted the fuzzy sets to get a better view of the situation. In this case the f-weighted possibilistic correlation of A and B is computed as (see [2] for details),

$$\rho_f(A,B) = \int_0^1 -\frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}.$$

Consider now the case when $A(x) = B(x) = x \cdot \chi_{[0,1]}(x)$ for $x \in \mathbb{R}$, that is $[A]^\gamma = [B]^\gamma = [\gamma, 1]$, for $\gamma \in [0,1]$. Suppose that their joint possibility distribution is given by

$$W(x,y) = \max\{x + y - 1, 0\}.$$

Then we get

$$\rho_f(A,B) = -\int_0^1 \frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}.$$

We note here that W is nothing else but the Lukasiewitz t-norm, or in the statistical literature, W is generally referred to as the lower Fréchet-Hoeffding bound for copulas.

2 A transition from zero to -1/2

Suppose that a family of joint possibility distribution of A and B (where $A(x) = B(x) = (1-x)\cdot\chi_{[0,1]}(x)$, for $x\in\mathbb{R}$) is defined by

$$C_n(x,y) = \begin{cases} 1 - x - \frac{n-1}{n}y, & \text{if } 0\le x\le 1, x\le y, \dfrac{n-1}{n}y + x \le 1 \\ 1 - \frac{n-1}{n}x - y, & \text{if } 0\le y\le 1, y\le x, \dfrac{n-1}{n}x + y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

In the following, for simplicity, we well write C instead of C_n. A γ-level set of C is computed by

$$[C]^\gamma = \left\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le \frac{n}{2n-1}(1-\gamma), 0\le y\le 1-\gamma-\frac{n-1}{n}x\right\}\bigcup$$

$$\left\{(x,y)\in\mathbb{R}^2 \mid \frac{n}{2n-1}(1-\gamma)\le x\le 1-\gamma, 0\le \frac{n-1}{n}y\le 1-\gamma-x\right\}.$$

The density function of a uniform distribution on $[C]^\gamma$ can be written as

$$f(x,y) = \begin{cases} \dfrac{1}{\int_{[C]^\gamma} dxdy}, & \text{if } (x,y)\in[C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{2n-1}{n(1-\gamma)^2}, & \text{if } (x,y)\in[C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the marginal density functions:

$$f_1(x) = \begin{cases} \dfrac{(2n-1)(1-\gamma-x)}{(n-1)(1-\gamma)^2}, & \text{if } \dfrac{n}{2n-1}(1-\gamma)\le x\le 1-\gamma \\ \dfrac{(2n-1)(1-\gamma-\frac{n-1}{n}x)}{n(1-\gamma)^2}, & \text{if } 0\le x\le \dfrac{n}{2n-1}(1-\gamma) \\ 0 & \text{otherwise} \end{cases}$$

and,

$$f_2(y) = \begin{cases} \dfrac{(2n-1)(1-\gamma-y)}{(n-1)(1-\gamma)^2}, & \text{if } \dfrac{n}{2n-1}(1-\gamma)\le y\le 1-\gamma \\ \dfrac{(2n-1)(1-\gamma-\frac{n-1}{n}y)}{n(1-\gamma)^2}, & \text{if } 0\le y\le \dfrac{n}{2n-1}(1-\gamma) \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables X_γ and Y_γ, whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$ as,

$$\begin{aligned} M(X_\gamma) &= \frac{2n-1}{n(1-\gamma)^2} \int_0^{\frac{n(1-\gamma)}{2n-1}} x(1-\gamma-\frac{n-1}{n}x)dx \\ &+ \frac{2n-1}{(n-1)(1-\gamma)^2} \int_{\frac{n(1-\gamma)}{2n-1}}^{1-\gamma} x(1-\gamma-x)dx = \frac{(1-\gamma)(4n-1)}{6(2n-1)} \end{aligned}$$

and, $M(Y_\gamma) = \dfrac{(1-\gamma)(4n-1)}{6(2n-1)}$.

(We can easily see that for $n = 1$ we have $M(X_\gamma) = \dfrac{1-\gamma}{2}$, and for $n \to \infty$ we find $M(X_\gamma) \to \dfrac{1-\gamma}{3}$.) We calculate the variations of X_γ and Y_γ as,

$$\begin{aligned} M(X_\gamma^2) &= \frac{2n-1}{n(1-\gamma)^2} \int_0^{\frac{n(1-\gamma)}{2n-1}} x^2(1-\gamma-\frac{n-1}{n}x)dx \\ &+ \frac{2n-1}{(n-1)(1-\gamma)^2} \int_{\frac{n(1-\gamma)}{2n-1}}^{1-\gamma} x^2(1-\gamma-x)dx \\ &= \frac{(1-\gamma)^2((2n-1)^3+8n^3-6n^2+n)}{12(2n-1)^3}. \end{aligned}$$

(We can easily see that for $n = 1$ we get $M(X_\gamma^2) = \dfrac{(1-\gamma)^2}{3}$, and for $n \to \infty$ we find $M(X_\gamma^2) \to \dfrac{(1-\gamma)^2}{6}$.) Furthermore,

$$\begin{aligned} \operatorname{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 &= \frac{(1-\gamma)^2((2n-1)^3+8n^3-6n^2+n)}{12(2n-1)^3} \\ &- \frac{(1-\gamma)^2(4n-1)^2}{36(2n-1)^2} = \frac{(1-\gamma)^2(2(2n-1)^2+n)}{36(2n-1)^2}. \end{aligned}$$

And similarly we obtain

$$\operatorname{var}(Y_\gamma) = \frac{(1-\gamma)^2(2(2n-1)^2+n)}{36(2n-1)^2}.$$

(We can easily see that for $n = 1$ we get $\operatorname{var}(X_\gamma) = \dfrac{(1-\gamma)^2}{12}$, and for $n \to \infty$ we find $\mathrm{var}(X_\gamma) \to \dfrac{(1-\gamma)^2}{18}$.) And,

$$\begin{aligned}
\mathrm{cov}(X_\gamma, Y_\gamma) &= M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) \\
&= \frac{(1-\gamma)^2 n(4n-1)}{12(2n-1)^2} - \frac{(1-\gamma)^2(1-n)(4n-1)}{36(2n-1)^2}.
\end{aligned}$$

(We can easily see that for $n = 1$ we have $\mathrm{cov}(X_\gamma, Y_\gamma) = 0$, and for $n \to \infty$ we find $\mathrm{cov}(X_\gamma, Y_\gamma) \to -\dfrac{(1-\gamma)^2}{36}$.) We can calculate the probabilisctic correlation of the random variables,

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}} = \frac{(1-n)(4n-1)}{2(2n-1)^2 + n}.$$

(We can easily see that for $n = 1$ we have $\rho(X_\gamma, Y_\gamma) = 0$, and for $n \to \infty$ we find $\rho(X_\gamma, Y_\gamma) \to -\dfrac{1}{2}$.) And finally the f-weighted possibilistic correlation of A and B is computed as,

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma = \frac{(1-n)(4n-1)}{2(2n-1)^2 + n}.$$

We obtain, that $\rho_f(A, B) = 0$ for $n = 1$ and if $n \to \infty$ then $\rho_f(A, B) \to -\dfrac{1}{2}$.

3 A transition from -1/2 to -3/5

Suppose that the joint possibility distribution of A and B (where $A(x) = B(x) = (1-x) \cdot \chi_{[0,1]}(x)$, for $x \in \mathbb{R}$) is defined by

$$C_n(x, y) = (1 - x - y) \cdot \chi_{T_n}(x, y),$$

where

$$T_n = \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1, \frac{1}{n-1} x \geq y \right\} \bigcup$$
$$\left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1, (n-1)x \leq y \right\}.$$

In the following, for simplicity, we well write C instead of C_n. A γ-level set of C is computed by

$$[C]^\gamma = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \frac{1}{n}(1-\gamma), (n-1)x \leq y \leq 1 - \gamma - x \right\} \bigcup$$

$$\left\{(x,y)\in\mathbb{R}^2 \mid 0\le y\le \frac{1}{n}(1-\gamma),(n-1)y\le x\le 1-\gamma-y\right\}.$$

The density function of a uniform distribution on $[C]^\gamma$ can be written as

$$f(x,y)=\begin{cases}\dfrac{1}{\int_{[C]^\gamma}dxdy}, & \text{if } (x,y)\in[C]^\gamma\\ 0 & \text{otherwise}\end{cases}=\begin{cases}\dfrac{n}{(1-\gamma)^2}, & \text{if } (x,y)\in[C]^\gamma\\ 0 & \text{otherwise}\end{cases}$$

We can calculate the marginal density functions:

$$f_1(x)=\begin{cases}\dfrac{n(1-\gamma-nx+\frac{x}{n-1})}{(1-\gamma)^2}, & \text{if } 0\le x\le\dfrac{1-\gamma}{n}\\ \dfrac{nx}{(1-\gamma)^2(n-1)}, & \text{if } \dfrac{(1-\gamma)}{n}\le x\le\dfrac{(n-1)(1-\gamma)}{n}\\ \dfrac{n(1-\gamma-x)}{(1-\gamma)^2}, & \text{if } \dfrac{(n-1)(1-\gamma)}{n}\le x\le 1-\gamma\\ 0 & \text{otherwise}\end{cases}$$

and,

$$f_2(y)=\begin{cases}\dfrac{n(1-\gamma-ny+\frac{y}{n-1})}{(1-\gamma)^2}, & \text{if } 0\le y\le\dfrac{1-\gamma}{n}\\ \dfrac{ny}{(1-\gamma)^2(n-1)}, & \text{if } \dfrac{(1-\gamma)}{n}\le y\le\dfrac{(n-1)(1-\gamma)}{n}\\ \dfrac{n(1-\gamma-y)}{(1-\gamma)^2}, & \text{if } \dfrac{(n-1)(1-\gamma)}{n}\le y\le 1-\gamma\\ 0 & \text{otherwise}\end{cases}$$

We can calculate the probabilistic expected values of the random variables X_γ and Y_γ, whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma\in[0,1]$ as

$$\begin{aligned}M(X_\gamma)=&\frac{n}{(1-\gamma)^2}\int_0^{\frac{1-\gamma}{n}}x(1-\gamma-nx+\frac{x}{n-1})dx\\ &+\frac{n}{(1-\gamma)^2}\int_{\frac{1-\gamma}{n}}^{\frac{(n-1)(1-\gamma)}{n}}\frac{x^2}{n-1}dx+\frac{n}{(1-\gamma)^2}\int_{\frac{(n-1)(1-\gamma)}{n}}^{1-\gamma)}x(1-\gamma-x)dx\\ =&\frac{1-\gamma}{3}.\end{aligned}$$

That is, $M(Y_\gamma) = \dfrac{1-\gamma}{3}$. We calculate the variations of X_γ and Y_γ as,

$$\begin{aligned}
M(X_\gamma^2) = \frac{n}{(1-\gamma)^2} \int_0^{\frac{1-\gamma}{n}} x^2(1-\gamma-nx+\frac{x}{n-1})dx \\
+ \frac{n}{(1-\gamma)^2} \int_{\frac{1-\gamma}{n}}^{\frac{(n-1)(1-\gamma)}{n}} \frac{x^3}{n-1}dx \\
+ \frac{n}{(1-\gamma)^2} \int_{\frac{(n-1)(1-\gamma)}{n}}^{1-\gamma)} x^2(1-\gamma-x)dx \\
= \frac{(1-\gamma)^2(3n^2-3n+2)}{12n^2}.
\end{aligned}$$

and,

$$\begin{aligned}
\mathrm{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1-\gamma)^2(3n^2-3n+2)}{12n^2} - \frac{(1-\gamma)^2}{9} \\
= \frac{(1-\gamma)^2(5n^2-9n+6)}{36n^2}.
\end{aligned}$$

And, similarly, we obtain

$$\mathrm{var}(Y_\gamma) = \frac{(1-\gamma)^2(5n^2-9n+6)}{36n^2}.$$

From

$$\mathrm{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = \frac{(1-\gamma)^2(3n-2)}{12n^2} - \frac{(1-\gamma)^2}{9},$$

we can calculate the probabilisctic correlation of the reandom variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}} = -\frac{-3n^2+7n-6}{5n^2-9n+6}.$$

And finally the f-weighted possibilistic correlation of A and B:

$$\rho_f(A,B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma = -\frac{-3n^2+7n-6}{5n^2-9n+6}$$

We obtain, that for $n = 2$

$$\rho_f(A,B) = -\frac{1}{2},$$

and if $n \to \infty$, then

$$\rho_f(A,B) \to -\frac{3}{5}.$$

We note that in this extremal case the joint possibility distribution is nothing else but the marginal distributions themselves, that is, $C_\infty(x,y) = 0$, for any interior point (x,y) of the unit square.

4 A trapezoidal case

Consider the case, when

$$A(x) = B(x) = \begin{cases} x, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } 1 \le x \le 2 \\ 3 - x, & \text{if } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

for $x \in \mathbb{R}$, that is $[A]^\gamma = [B]^\gamma = [\gamma, 3-\gamma]$, for $\gamma \in [0,1]$. Suppose that their joint possibility distribution is given by:

$$C(x,y) = \begin{cases} y, & \text{if } 0 \le x \le 3, 0 \le y \le 1, x \le y, x \le 3-y \\ 1, & \text{if } 1 \le x \le 2, 1 \le y \le 2, x \le y \\ x, & \text{if } 0 \le x \le 1, 0 \le y \le 3, y \le x, x \le 3-y \\ 0, & \text{otherwise} \end{cases}$$

Then $[C]^\gamma = \big\{(x,y) \in \mathbb{R}^2 \mid \gamma \le x \le 3-\gamma, \gamma \le y \le 3-x\big\}$.
The density function of a uniform distribution on $[F]^\gamma$ can be written as

$$f(x,y) = \begin{cases} \dfrac{1}{\int_{[C]^\gamma} dxdy}, & \text{if } (x,y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{2}{(3-2\gamma)^2}, & \text{if } (x,y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \dfrac{2(3-\gamma-x)}{(3-2\gamma)^2}, & \text{if } \gamma \le x \le 3-\gamma \\ 0 & \text{otherwise} \end{cases}$$

and,

$$f_2(y) = \begin{cases} \dfrac{2(3-\gamma-y)}{(3-2\gamma)^2}, & \text{if } \gamma \le y \le 3-\gamma \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables X_γ and Y_γ, whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$:

$$M(X_\gamma) = \frac{2}{(3-2\gamma)^2} \int_\gamma^{3-\gamma} x(3-\gamma-x)dx = \frac{\gamma+3}{3}$$

and,

$$M(Y_\gamma) = \frac{2}{(3-2\gamma)^2} \int_\gamma^{3-\gamma} y(3-\gamma-y)dx = \frac{\gamma+3}{3}$$

We calculate the variations of X_γ and Y_γ from the formula

$$\mathrm{var}(X) = M(X^2) - M(X)^2$$

as

$$M(X_\gamma^2) = \frac{2}{(3-2\gamma)^2}\int_\gamma^{3-\gamma} x^2(3-\gamma-x)dx = \frac{2\gamma^2+9}{6}$$

and,

$$\mathrm{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{2\gamma^2+9}{6} - \frac{(\gamma+3)^2}{9} = \frac{(3-2\gamma)^2}{18}.$$

And similarly we obtain

$$\mathrm{var}(Y_\gamma) = \frac{(3-2\gamma)^2}{18}.$$

Using the relationship,

$$\mathrm{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(3-2\gamma)^2}{36},$$

we can calculate the probabilisctic correlation of the random variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}} = -\frac{1}{2}.$$

And finally the f-weighted possibilistic correlation of A and B is,

$$\rho_f(A,B) = -\int_0^1 \frac{1}{2} f(\gamma)d\gamma = -\frac{1}{2}.$$

5 Time Series With Fuzzy Data

A time series with fuzzy data is referred to as fuzzy time series (see [3]). Consider a fuzzy time series indexed by $t \in (0,1]$,

$$A_t(x) = \begin{cases} 1 - \dfrac{x}{t}, & \text{if } 0 \le x \le t \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad A_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that in this case,

$$[A_t]^\gamma = [0, t(1-\gamma)],\ \gamma \in [0,1].$$

If we have $t_1, t_2 \in [0,1]$, then the joint possibility distribution of the corresponding fuzzy numbers is given by:

$$C(x,y) = \left(1 - \frac{x}{t_1} - \frac{y}{t_2}\right) \cdot \chi_T(x,y),$$

where

$$T = \left\{ (x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \frac{x}{t_1} + \frac{y}{t_2} \leq 1 \right\}.$$

Then $[C]^\gamma = \left\{ (x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \dfrac{x}{t_1} + \dfrac{y}{t_2} \leq 1 - \gamma \right\}$.

The density function of a uniform distribution on $[C]^\gamma$ can be written as

$$f(x,y) = \begin{cases} \dfrac{1}{\int_{[C]^\gamma} dxdy}, & \text{if } (x,y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

That is,

$$f(x,y) = \begin{cases} \dfrac{2}{t_1 t_2 (1-\gamma)^2}, & \text{if } (x,y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \dfrac{2(1-\gamma-\frac{x}{t_1})}{t_1(1-\gamma)^2}, & \text{if } 0 \leq x \leq t_1(1-\gamma) \\ 0 & \text{otherwise} \end{cases}$$

and,

$$f_2(y) = \begin{cases} \dfrac{2(1-\gamma-\frac{y}{t_2})}{t_2(1-\gamma)^2}, & \text{if } 0 \leq y \leq t_2(1-\gamma) \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables X_γ and Y_γ, whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$:

$$M(X_\gamma) = \frac{2}{t_1(1-\gamma)^2} \int_0^{t_1(1-\gamma)} x(1-\gamma-\frac{x}{t_1})dx = \frac{t_1(1-\gamma)}{3}$$

and

$$M(Y_\gamma) = \frac{2}{t_2(1-\gamma)^2} \int_0^{t_2(1-\gamma)} y(1-\gamma-\frac{y}{t_2})dx = \frac{t_2(1-\gamma)}{3}.$$

We calculate now the variations of X_γ and Y_γ as,

$$M(X_\gamma^2) = \frac{2}{t_1(1-\gamma)^2} \int_0^{t_1(1-\gamma)} x^2(1-\gamma-\frac{x}{t_1})dx = \frac{t_1^2(1-\gamma)^2}{6}$$

and,

$$\mathrm{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{t_1^2(1-\gamma)^2}{6} - \frac{t_1^2(1-\gamma)^2}{9} = \frac{t_1^2(1-\gamma)^2}{18}.$$

And, in a similar way, we obtain,

$$\mathrm{var}(Y_\gamma) = \frac{t_2^2(1-\gamma)^2}{18}.$$

From,

$$\mathrm{cov}(X_\gamma, Y_\gamma) = -\frac{t_1 t_2(1-\gamma)^2}{36},$$

we can calculate the probabilisctic correlation of the reandom variables,

$$\rho(X_\gamma, Y_\gamma) = \frac{\mathrm{cov}(X_\gamma, Y_\gamma)}{\sqrt{\mathrm{var}(X_\gamma)}\sqrt{\mathrm{var}(Y_\gamma)}} = -\frac{1}{2}.$$

The f-weighted possibilistic correlation of A_{t_1} and A_{t_2},

$$\rho_f(A_{t_1}, A_{t_2}) = \int_0^1 -\frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}.$$

So, the autocorrelation function of this fuzzy time series is constant. Namely,

$$R(t_1, t_2) = -\frac{1}{2},$$

for all $t_1, t_2 \in [0, 1]$.

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