On the Structure of Finite Involutive Uninorm Chains

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Abstract: We will give a state of the art summary on the structural description of involutive uninorm algebras.

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1 Results

Involutive uninorm algebras are (not necessarily integral) commutative residuated lattices with an element *f* which defines an involution. In more detail:

Definition 1 $\mathcal{U} = \langle X, \bullet, \leq, \bot, \top, e, f, \rangle$ is called an *involutive uninorm algebra* if

1. $\mathcal{C} = \langle X, \leq, \bot, \top \rangle$ is a bounded poset,
2. $\bullet$ is a uninorm over $\mathcal{C}$ with neutral element e,
3. for every $x \in X$, $x \rightarrow_\bullet f = \max\{z \in X \mid x \bullet z \leq f\}$ exists, and
4. for every $x \in X$, we have $(x \rightarrow_\bullet f) \rightarrow_\bullet f = x$.

It is not difficult to see that every involutive uninorm is residuated (see [10]) and hence $\bullet$ is isotone (see [6]). Therefore, $' : X \rightarrow X$ given by

$$x' = x \rightarrow_\bullet f$$

is an order-reversing involution.

If $\mathcal{C}$ is linearly ordered, we call $\mathcal{U}$ an involutive uninorm *chain*. $\mathcal{U}$ is called *finite* if X is a finite set.

By using the concept of skew pairs a structural description has been given for the case when *e=f* and the underlying universe of the involutive uninorm algebra is a complete and densely ordered chain [10]. In this paper we present some results for the finite chain case.

For finite uninorm chains we define a new concept, the rank of the algebra as follows:

Definition 2 Consider a finite involutive uninorm chain $\mathcal{U}$ and denote the cardinality of its universe by n. Clearly, $\mathcal{U}$ is order-isomorphic to a finite involutive uninorm chain with universe $\{1, 2, \ldots, n\} \subset \mathbf{N}$, denote it by $\langle\{1, 2, \ldots, n\}, \circledast, \leq, 1, n, e, f\rangle$. Call $e - f$ the *rank* of $\mathcal{U}$. It is easy to see that the rank is well-defined.

Standing assumption:
Because of the order-isomorphism which was mentioned in Definition 2, without loss of generality, in the sequel we will consider finite involutive uninorm chains *solely* on the universe $\{1, 2, \ldots, n\}$,[2] and will employ the shorter notation

$$\mathcal{U}_n = \langle\{1, 2, \ldots, n\}, \circledast, \leq, e, f\rangle.^3$$

We have the following structural description.

Definition 3 For any involutive uninorm algebra $\mathcal{U} = \langle X, \circledast, \leq, \bot, \top, e, f, \rangle$ define

$$X^+ = \{x \in X \mid x \geq e\} \qquad \text{and} \qquad X^- = \{x \in X \mid x \leq e\}.$$

Proposition 2 *Let $\langle X, \circledast, \leq, \bot, \top, e, f, \rangle$ be an involutive uninorm algebra, $\otimes$ its underlying t-norm and $\oplus$ its underlying t-conorm acting on X^+ and X^-, respectively. Then $\otimes$ and $\oplus$ uniquely determine $\circledast$ on $X^+ \times X^-$ via*

$$x \circledast y = \begin{cases} (x \to_\oplus y')', & \text{if } x \leq y' \\ (y \to_\otimes x')', & \text{if } x > y' \end{cases} \tag{5}$$

Corollary 1 *If there are no elements in X which are incomparable with e in an involutive uninorm algebra $\langle X, \circledast, \leq, \bot, \top, e, f, \rangle$ then the underlying t-norm and t-conorm of $\circledast$ uniquely determine $\circledast$.*

This structural description motivates the following construction.

Definition 5 Let $\otimes$ be a t-norm on $\{1, 2, \ldots, e\}$, $\oplus$ be a t-conorm on $\{e, e+1, \ldots, n\}$, and let $x' = n + 1 - x$ for $x \in \{1, 2, \ldots, n\}$. Denote

$$\mathcal{U}_\otimes^\oplus = \langle\{1, 2, \ldots, n\}, \circledast, \leq, e, f\rangle$$

where

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \leq e \\ x \oplus y & \text{if } x, y \geq e \\ (x \to_\oplus y')' & \text{if } (x \geq e,\ y \leq e, \text{ and } x \leq y') \text{ or } (y \geq e,\ x \leq e, \text{ and } x \leq y') \\ (y \to_\otimes x')' & \text{if } (x \geq e,\ y \leq e, \text{ and } x > y') \text{ or } (y \geq e,\ x \leq e, \text{ and } x > y') \end{cases} \tag{9}$$

Consider a finite involutive uninorm chain $\mathcal{U}_n = \langle\{1, 2, \ldots, n\}, \circledast, \leq, e, f\rangle$ and denote its underlying t-norm (which acts on $\{1, 2, \ldots, e\}$) and its underlying t-conorm (which acts on $\{e, e+1, \ldots, n\}$) by $\otimes$ and $\oplus$, respectively. By Corollary 1 we have $\mathcal{U}_n = \mathcal{U}_\otimes^\oplus$.

Call an involutive uninorm $\top\bot$-*indecomposable* if [2,n-1] (that is, we remove top and bottom from the underlying universe) is not a subalgebra of it.

The following two theorems hold true.

Theorem 1 *We have that $\circledast$ is the monoidal operation of a finite involutive uninorm chain with rank $= 0$ (resp. rank $= 1$) iff n is odd (resp. n is even) and*

$$x \circledast y = \begin{cases} \min(x, y) & \text{if } x \leq y' \\ \max(x, y), & \text{if } x > y' \end{cases} \tag{8}$$

Theorem 2 *There is a one-to-one correspondence between* $\top\bot$*-indecomposable involutive uninorms with rank 2 on n-element chains and conorm operations on* $\frac{n-1}{2}$*-element chains given as follows:*
Let $\odot$ *be the t-norm operation on* $\{1, 2, \ldots, \frac{n+3}{2}\}$ *given by*

$$x \odot y = \begin{cases} 1 & \text{if } x, y < \frac{n+3}{2} \\ \min(x, y) & \text{otherwise} \end{cases} . \quad (13)$$

1. *For any involutive uninorm on* $\{1, \ldots, n\}$ *with rank* $= 2$*, its underlying t-norm is equal to* $\odot$*.*
2. *For any conorm operation* $\oplus$ *on* $\{\frac{n+3}{2}, \frac{n+3}{2}+1, \ldots, n\}$*, the monoidal operation of* $\mathcal{U}_{\odot}^{\oplus}$ *is an involutive uninorm on* $\{1, \ldots, n\}$ *with rank* $= 2$*.*

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References

[1] G. Birkhoff, Lattice Theory, Amer. Math Soc. Colloquium Publications, third edition (Amer. Math. Soc., RI), 1973.

[2] J. C. Fodor, R. R. Yager, A. Rybalov, *Structure of uninorms*, International Journal Of Uncertainty Fuzziness And Knowledge-Based Systems, 5 (1997), 411–427.

[3] L. Fuchs, Partially Ordered Algebraic Systems, *Pergamon Press, Oxford-London-New York-Paris* (1963).

[4] D. Gabbay, G. Metcalfe, *Fuzzy logics based on* $[0, 1[$*-continuous uninorms*, Archive for Mathematical Logic, 46(6): 425–469, 2007.

[5] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated Lattices – An Algebraic Glimpse at Substructural Logics, *Elsevier*, 2007, 532 pp.

[6] U. Höhle, *Commutative residuated l-monoids*, in: Topological and Algebraic Structures in Fuzzy Sets, A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, (E.P. Klement, S. E. Rodabaugh, eds.), Trends in Logic, vol 20. Kluwer Academic Publishers, Dordrecht, 2003, 53–106.

[7] S. Jenei, *On reflection invariance of residuated chains*, Annals of Pure and Applied Logic, Volume 161, Issue 2, November 2009, Pages 220–227.

[8] S. Jenei, *On the structure of rotation-invariant semigroups*, Archive for Mathematical Logic, 42 (2003), 489–514.

[9] S. Jenei, *On the relationship between the rotation construction and ordered abelian groups*, Fuzzy Sets and Systems (in press)

[10] S. Jenei, *Structural description of a class of involutive uninorms via skew symmetrization*, Journal of Logic and Computation, doi:10.1093/logcom/exp060

[11] S. Jenei, F. Montagna, *On the continuity points of left-continuous t-norms*, Archive for Mathematical Logic, 42 (2003), 797–810.

[12] K.C. Maes, B. De Baets, *On the structure of left-continuous t-norms that have a continuous contour line*, Fuzzy Sets and Systems 158 (2007), 843–860.

[13] G. Metcalfe, F. Montagna, *Substructural fuzzy logics*, Journal of Symbolic Logic, 72(3): 834–864, 2007.

[14] R. R. Yager, A. Rybalov, *Uninorm aggregation operators*, Fuzzy Sets and Systems, Vol. 80 (1996), 111–120.